Truncation identities for the small polaron fusion hierarchy

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We study a one-dimensional lattice model of interacting spinless fermions. This model is integrable for both periodic and open boundary conditions, the latter case includes the presence of Grassmann valued non-diagonal boundary fields breaking the bulk $U(1)$ symmetry of the model. Starting from the embedding of this model into a graded Yang-Baxter algebra an infinite hierarchy of comuting transfer matrices is constructed by means of a fusion procedure. For certain values of the coupling constant related to anisotropies of the underlying vertex model taken at roots of unity this hierarchy is shown to truncate giving a finite set of functional equations for the spectrum of the transfer matrices. For generic coupling constants the spectral problem is formulated in terms of a TQ-equation which can be solved by Bethe ansatz methods for periodic and diagonal open boundary conditions. Possible approaches for the solution of the model with generic non-diagonal boundary fields are discussed.

I. INTRODUCTION

The small polaron model provides an effective description of the behaviour of an additional electron in a polar crystal [1, 2]. In one spatial dimension this lattice system of interacting spinless fermions can be constructed within the framework of the Quantum Inverse Scattering Method [3] allowing to compute the excitation spectrum by Bethe ansatz techniques, see e.g. [4, 5]. By means of a graded generalization [6–8] of Sklyanin's reflection algebra [9] it was possible to provide the small polaron model with open boundary conditions while keeping its integrability intact. These integrable boundary conditions are encoded in c-number valued 2×2-matrix solutions to the reflection equations [10–12].

Diagonal boundary matrices correspond to boundary chemical potentials in the Hamiltonian. In this case the spectrum of the open small polaron model can be studied again using Bethe ansatz methods [13–15]. As a consequence of the underlying grading the general non-diagonal solutions to the reflection equations are *super matrices* [16]. The resulting additional boundary terms do not conserve particle number and have anti-commuting scalars, i.e. odd Grassmann numbers, as amplitudes. The fact that the $U(1)$ symmetry of the model is broken implies that there is no simple eigenstate (e.g. the Fock vacuum) of the model which can be used as a reference state for

the algebraic Bethe ansatz. This situation is, in fact, very similar to the case of non-diagonal boundary magnetic fields in the closely related spin-1/2 Heisenberg chains: in all approaches used so far the solution of the spectral problem is limited to small finite systems, with constraints between the boundary fields at the two ends of the chain or restrictions on the anisotropy limiting their usefulness in the thermodynamic limit [17–25].

In a previous publication [26] we have investigated the applicability of Bethe ansatz methods in the simpler case of a model of free fermions with similar open boundary conditions. We found that for a certain class of non-diagonal boundary super matrices, a unitary transformation on the auxiliary space allowed for an exact solution of the free fermion model. Furthermore, the functional equations obtained there could be easily generalized to describe the spectrum of the model for arbitrary non-diagonal boundary fields. Unfortunatley, this approach can not be applied to the small polaron model.

Following ideas [17, 20] used in the context of the spin-1/2 XXZ chain before we adapt the fusion procedure for the transfer matrix of the quantum chain to the graded case of the small polaron model. Assuming the existence of a certain limit we obtain a system of functional TQ-equations for the eigenvalues of the transfer matrix which are shown to coincide with the result obtained from the algebraic Bethe ansatz for both periodic and diagonal open boundary conditions. For special values of the interaction parameter related to roots of unity of the anisotropy parameter we derive truncation identities for the fusion of the relevant objects, in particular the transfer matrices. Using these identities the TQ-equation reduces to a a set of relations between finitely many quantities.

II. THE SMALL POLARON AS A FUNDAMENTAL INTEGRABLE MODEL

Some materials exhibit a strong electron-phonon coupling that considerably reduces the mobility of electrons within the conduction band. This interaction may be regarded as an increase of the electron's effective mass, thus giving raise to quasi-particles called *polarons*. If the electron is essentially trapped at a single lattice-site the corresponding quasi-particle is said to be a *small polaron*. In this case, electron transport occurs either by thermally activated hopping (at high temperatures) or by tunneling (at low temperatures).

In the case of periodic boundary conditions (PBC) the N-site small polaron model is characterized by the Hamiltonian

$$H^{\text{PBC}} = \sum_{j=1}^{N} H_{j,j+1} \qquad \text{with} \qquad H_{N,N+1} \equiv H_{N,1} \tag{2.1}$$

with a Hamiltonian density $H_{j,j+1}$ defined as

$$H_{j,j+1} = -t\left(c_{j+1}^\dagger c_j + c_j^\dagger c_{j+1}\right) + V\left(n_{j+1}n_j + \bar{n}_{j+1}\bar{n}_j\right) \tag{2.2}$$

where $c_k^\dagger$ and c_k label the creation resp. annihilation operators of spinless fermions at site k, which are subject to the anticommutation relations $[c_\ell^\dagger, c_k]_+ = \delta_{\ell k}$. Moreover, it is convenient to define number operators $n_k \equiv c_k^\dagger c_k = 1 - \bar{n}_k$. In this context, the parameters t and V may be interpreted as hopping amplitude and density-density interaction strength respectively.

A. Construction within the QISM framework

The small polaron model can be associated to a graded six-vertex model with anisotropy η and R-matrix

$$R(u) = \frac{1}{\sin(2\eta)}\begin{pmatrix} \sin(u+2\eta) & 0 & 0 & 0 \\ 0 & \sin(u) & \sin(2\eta) & 0 \\ 0 & \sin(2\eta) & \sin(u) & 0 \\ 0 & 0 & 0 & -\sin(u+2\eta) \end{pmatrix}. \tag{2.3}$$

$R(u)$ a solution to the Yang-Baxter Equation (YBE)

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v) \tag{2.4}$$

and enjoys several useful properties, such as

- *P-symmetry*

$$R_{21}(u) \equiv \mathcal{P}_{12}R_{12}(u)\mathcal{P}_{12} = R_{12}(u) \tag{2.5a}$$

- *T-symmetry*

$$R_{12}^{\mathrm{st}_1\mathrm{st}_2}(u) = R_{12}^{\mathrm{ist}_1\mathrm{ist}_2}(u) = R_{21}(u) \tag{2.5b}$$

- *regularity*

$$R_{12}(0) = \mathcal{P}_{12} \tag{2.5c}$$

- *unitarity*

$$R_{12}(u)R_{21}(-u) = \zeta(u) \tag{2.5d}$$

where the scalar function $\zeta(u)$ is given by

$$\zeta(u) \equiv g(u)g(-u) \qquad \text{and} \qquad g(u) \equiv -\frac{\sin(u-2\eta)}{\sin(2\eta)} \ .$$

Unitarity of an R-matrix is of course a direct consequence of its regularity.

- *crossing symmetry*

$$R_{21}^{\mathrm{st}_2}(-u-4\eta)R_{21}^{\mathrm{st}_1}(u) = \zeta(u+2\eta) \tag{2.5e}$$

- *periodicity*

$$R_{12}(u+\pi) = -\sigma_2^z\ R_{12}(u)\ \sigma_2^z = -\sigma_1^z\ R_{12}(u)\ \sigma_1^z \tag{2.5f}$$

The periodicity $R(u+2\pi) = R(u)$ is obvious from definition (2.3).

The operations of partial super transposition st_a and inverse partial super transposition ist_a as well as the graded permutation operator $\mathcal{P}_{ab}$ and the notion of super tensor product structures are explained in appendix A. Unless stated otherwise, all embeddings are to be understood in a *graded* sense, that is into a super tensor product structure. Considering the Yang-Baxter Algebra (YBA)

$$R_{12}(u-v)\ T_1(u)\ T_2(v) = T_2(v)\ T_1(u)\ R_{12}(u-v) \tag{2.6}$$

this means that $T_1(u) \equiv T(u) \otimes_s \mathbb{1}$ and $T_2(v) \equiv \mathbb{1} \otimes_s T(v)$.

The small polaron model constructed here is fundamental, i.e. the Lax-operators $L_j(u)$, being local solutions to (2.6) are just graded embeddings of the above R-Matrix (2.3),

$$L_j(u) = \frac{1}{\sin(2\eta)} \begin{pmatrix} \sin(u)n_j + \sin(u+2\eta)\bar{n}_j & \sin(2\eta)c_j^\dagger \\ \sin(2\eta)c_j & \sin(u)\bar{n}_j - \sin(u+2\eta)n_j \end{pmatrix} . \tag{2.7}$$

As a consequence of the YBA's co-multiplication property, a specific global representation, the so-called monodromy matrix, can be constructed as a product of Lax-operators taken in auxiliary space,

$$T(u) \equiv L_N(u) \cdot \ldots \cdot L_2(u) \cdot L_1(u) \tag{2.8}$$

and gives rise to a family of commuting (super) transfer matrices

$$\tau(u) \equiv \mathrm{str}\,\{\,T(u)\,\} \quad \Rightarrow \quad [\tau(u), \tau(v)] = 0 \quad \forall u, v \in \mathbb{C}\ , \tag{2.9}$$

where $\mathrm{str}\,\{\,\cdot\,\}$ denotes the supertrace defined in appendix A. In particular, the PBC hamiltonian (2.1) with $t = 1$ and $V = -\cos(2\eta)$ is among these commuting operators,

$$H^{PBC} = -\sin(2\eta)\frac{\mathrm{d}}{\mathrm{d}u}\ln\tau(u)\bigg|_{u=0} \ . \tag{2.10}$$

B. Asymptotic behaviour of the PBC transfer matrix

By construction the monodromy matrix (and similarly the transfer matrix) is a Laurent polynomial in $z \equiv \mathrm{e}^{\mathrm{i}u}$, i.e. $T(u) = \sum_{k=-N}^{N} T_k z^k$. For $z \to \infty$ the Lax-operators (2.7) are

$$L_j(u) \approx \frac{z}{2\mathrm{i}\sin(2\eta)} \begin{pmatrix} n_j + \mathrm{e}^{2\mathrm{i}\eta}\ \bar{n}_j & 0 \\ 0 & \bar{n}_j - \mathrm{e}^{2\mathrm{i}\eta}\ n_j \end{pmatrix} \tag{2.11}$$

and consequently the asymptotic behaviour of the (super) transfer matrix is given by

$$\tau(u) \approx \left(\frac{z}{2\mathrm{i}\sin(2\eta)}\right)^N \mathrm{e}^{\mathrm{i}N\eta} \left[\prod_{j=1}^{N} \left(\mathrm{e}^{-\mathrm{i}\eta} n_j + \mathrm{e}^{\mathrm{i}\eta}\bar{n}_j\right) - \prod_{j=1}^{N} \left(\mathrm{e}^{-\mathrm{i}\eta}\bar{n}_j - \mathrm{e}^{\mathrm{i}\eta} n_j\right)\right] . \tag{2.12}$$

As the leading term comprises only diagonal operators the first order contributions to the transfer matrix eigenvalues $\Lambda_M(u)$ can easily be determined and are found to depend on the total number of particles M,

$$\Lambda_M(u) \approx \mathrm{e}^{\mathrm{i}uN} \left(\frac{\mathrm{e}^{\mathrm{i}\eta}}{\mathrm{e}^{\mathrm{i}\,2\eta} - \mathrm{e}^{-\mathrm{i}\,2\eta}}\right)^N \left(\mathrm{e}^{\mathrm{i}N\eta}\mathrm{e}^{-\mathrm{i}M2\eta} - (-1)^M \mathrm{e}^{-\mathrm{i}N\eta}\mathrm{e}^{\mathrm{i}M2\eta}\right) . \tag{2.13}$$

This result will be used to fix the degree of the Q-functions in section IV.

III. FUSION OF THE R-MATRIX IN AUXILIARY SPACE

Given an R-matrix as solution to the YBE (2.4), the fusion procedure allows for the construction of *larger* R-matrices as solutions to corresponding Yang-Baxter equations, where larger refers to the dimensionality of the auxiliary space involved. All that fusion requires is a pair of complementary orthogonal[1] projectors P_{12}^+ and P_{12}^- such that for a specific value of $\rho \in \mathbb{C}$ the following *triangularity condition* holds for arbitrary spectral parameters $u \in \mathbb{C}$,

$$P_{12}^-\ R_{13}(u)\ R_{23}(u+\rho)\ P_{12}^+ = 0\ . \tag{3.1}$$

By virtue of this condition, it can be shown that the *fused* R-Matrix, defined by

$$R_{(12)3}(u) \equiv P_{12}^+ R_{13}(u) R_{23}(u+\rho) P_{12}^+ \,. \tag{3.2}$$

satisfies the corresponding Yang-Baxter equation

$$R_{(12)3}(u-v)\ R_{(12)4}(u)\ R_{34}(v) = R_{34}(v)\ R_{(12)4}(u)\ R_{(12)3}(u-v)\ . \tag{3.3}$$

[1] As usual, *orthogonal* means $P_{12}^+\ P_{12}^- = 0$ whereas *complementary* refers to the property $P_{12}^+ + P_{12}^- = \mathbb{1}$.

It is easily found that the small polaron R-matrix (2.3) has two distinct singularities at $u = \pm 2\eta$,

$$\det\{R(u)\} = -\frac{\sin(u-2\eta)}{\sin(2\eta)}\left(\frac{\sin(u+2\eta)}{\sin(2\eta)}\right)^3 \overset{!}{=} 0\,. \tag{3.4}$$

At $u = -2\eta$ the R-Matrix gives rise to a projector onto a one-dimensional subspace,

$$P^- \equiv -\frac{1}{2}R(-2\eta) = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\,. \tag{3.5}$$

However, unlike in the case of the Heisenberg spin chain, the orthogonal projector P^+ onto the complementary three-dimensional subspace cannot be obtained from the R-matrix at the second singularity,

$$P^+ \equiv \mathbb{1} - P^- \neq \frac{1}{2}R(2\eta)\,. \tag{3.6}$$

Using this projector, fusion of two small polaron R-matrices in the auxiliary space can be achieved by means of (3.2) with $\rho = 2\eta$,

$$R_{(12)3}(u) \equiv P^+_{12}R_{13}(u)R_{23}(u+2\eta)P^+_{12}\,. \tag{3.7}$$

The resulting object $R_{(12)3}(u)$ is an 8×8-matrix of rank 6 and may therefore be effectively reduced to a 6×6-matrix $R_{\ll 12\gg 3}(u)$ acting on a three-dimensional auxiliary space $V_{\ll 12\gg}$ and on a two-dimensional quantum space V_3. Changing from the $BFFB$-graded[2] canonical basis

$$\mathcal{B}_0 = \{e_1,\, e_2,\, e_3,\, e_4\}_{BFFB} \equiv \{|0\rangle\otimes|0\rangle,\, |0\rangle\otimes|1\rangle,\, |1\rangle\otimes|0\rangle,\, |1\rangle\otimes|1\rangle\}_{BFFB} \tag{3.8}$$

to the projectors' $BFBF$-graded singlet/triplet-eigenbasis

$$\mathcal{B}_\pm = \{f_1,\, f_2,\, f_3,\, f_4\}_{BFBF} \equiv \{e_1,\, \frac{e_2+e_3}{\sqrt{2}},\, e_4,\, \frac{e_2-e_3}{\sqrt{2}}\}_{BFBF}\,. \tag{3.9}$$

the matrix $R_{(12)3}(u)$ gains the advantageous shape

$$\left(\begin{array}{c|cc} R_{\ll 12\gg 3}(u) & & \\ \hline & 0 & 0 \\ & 0 & 0 \end{array}\right)^{i}_{j} = (f_i)^T\left[R_{(12)3}(u)\right] f_j \tag{3.10}$$

[2] This notation is explained in appendix A.

where $R_{\ll 12\gg 3}(u)$ is the only non-vanishing block. Explicitly one finds,

$$R_{\ll 12\gg 3}(u) \propto \left(\begin{array}{cc|cc|cc} 2\sin(u+4\eta) & 0 & 0 & 0 & 0 & 0 \\ 0 & 2\sin(u) & \sqrt{2}\sin(4\eta) & 0 & 0 & 0 \\ \hline 0 & 2\sqrt{2}\sin(2\eta) & 2\sin(u+2\eta) & 0 & 0 & 0 \\ 0 & 0 & 0 & -2\sin(u+2\eta) & -2\sqrt{2}\sin(2\eta) & 0 \\ \hline 0 & 0 & 0 & \sqrt{2}\sin(4\eta) & 2\sin(u) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\sin(u+4\eta) \end{array}\right) . \tag{3.11}$$

A. General construction of higher fused R-matrices

In general, higher fused R-matrices can be constructed employing the projection operators

$$P^{+}_{1\dots n} \equiv \frac{1}{n!} \sum_{\sigma\in\mathcal{S}_n} P_\sigma \; . \tag{3.12}$$

Here σ runs through all the elements of the permutation group $\mathcal{S}_n$ and P_σ is the permutation operator corresponding to σ. Now the higher fused R-matrices are obtained as

$$R_{(1\dots n)q}(u) \equiv P^{+}_{1\dots n}\; R_{1q}(u)\; R_{2q}(u+2\eta)\; \dots\; R_{nq}(u+[n-1]\cdot 2\eta) P^{+}_{1\dots n} \tag{3.13}$$

Just as for the first fusion step, it is convenient to apply a similarity transformation $A_{(1\dots n)}$ into the eigenbasis[3] of the projection operators,

$$A_{(1\dots n)} R_{(1\dots n)q}(u) A^{-1}_{(1\dots n)} \equiv \left(\begin{array}{c|c} R_{\ll 1\dots n\gg q}(u) & \\ \hline & \begin{matrix} 0 & \\ & \ddots \end{matrix} \end{array}\right) . \tag{3.14}$$

The first few ($n = 1, 2, 3, 4$) transformation matrices $A_{(1\dots n)}$ are explicitly given in appendix E. By construction, all matrix elements of (3.14), except for those in the upper left $2(n+1)\times 2(n+1)$ block, vanish. This block is referred to as the fused R-matrix $R_{\ll 1\dots n\gg q}(u)$. As shown in table I, its fused auxiliary space has alternating gradation (bosonic, fermionic, . . .).

auxiliary space:	$\ll 12\gg$	$\ll 123\gg$	$\ll 1234\gg$	$\ll 12345\gg$	. . .
grading:	BFB	$BFBF$	$BFBFB$	$BFBFBF$	. . .

[3] Since the projectors here are just the same as for the XXZ Heisenberg spin chain, the respective transformation is simply given by the matrix of Clebsch-Gordan coefficients.

TABLE I: Gradation of the fused auxiliary spaces in the projector eigenbasis.

The periodicity property (2.5f) carries over to the fused R-matrices,

$$R_{\ll 1\ldots n \gg q}(u+\pi) = (-1)^n \; \sigma^z_{\ll n \gg} \; R_{\ll 1\ldots n \gg q}(u) \; \sigma^z_{\ll n \gg} \tag{3.15}$$

with $\sigma^z_{\ll n \gg}$ being defined through

$$\sigma^z_{(n)} \equiv \prod_{k=1}^{n} \sigma^z_k \qquad \text{and} \qquad A_{(12\ldots n)} \; \sigma^z_{(n)} \; A^{-1}_{(12\ldots n)} \equiv \left(\begin{array}{c|c} \sigma^z_{\ll n \gg} & \\ \hline & \begin{array}{cc} * & \\ & \ddots \end{array} \end{array}\right) . \tag{3.16}$$

B. Fusion hierarchy for super transfer matrices

Since, by construction, the fused R-matrices again satisfy the YBE they can be used to establish further families of commuting operators as supertraces of fused monodromy matrices,

$$\begin{aligned} T_{(12\ldots n)}(u) \;&\equiv\; P^+_{12\ldots n} \, R_{(12\ldots n)q_N}(u) \; \cdot \ldots \cdot \; R_{(12\ldots n)q_2}(u) \; R_{(12\ldots n)q_1}(u) \; P^+_{12\ldots n} \\ &=\; P^+_{12\ldots n} \; T_{(12\ldots n-1)}(u) \; T_n(u+[n-1]\cdot 2\eta) \; P^+_{12\ldots n} \, , \end{aligned} \tag{3.17}$$

$$A_{(12\ldots n)} \; T_{(12\ldots n)}(u) \; A^{-1}_{(12\ldots n)} \equiv \left(\begin{array}{c|c} T_{\ll 12\ldots n \gg}(u) & \\ \hline & \begin{array}{cc} 0 & \\ & \ddots \end{array} \end{array}\right) . \tag{3.18}$$

Indeed it is found that the (super) transfer matrices obtained from any fusion level n,

$$\tau^{(n)}(u) \equiv \mathrm{str}_{(12\ldots n+1)} \left\{ T_{(12\ldots n+1)}(u) \right\} = \mathrm{str}_{\ll 12\ldots n+1 \gg} \left\{ T_{\ll 12\ldots n+1 \gg}(u) \right\} \; , \tag{3.19}$$

commute with the transfer matrices of any other fusion level m, i.e. $[\tau^{(n)}(u), \tau^{(m)}(v)] = 0$ for all $u, v \in \mathbb{C}$ and arbitrary $n, m \in \mathbb{N}_0$. A most interesting fact is, that these *fused* transfer matrices obey certain functional relations, known as *fusion hierarchy.* For the periodic boundary case, the fusion hierarchy reads,

$$\begin{aligned} \tau^{(0)}(u) \; \tau^{(0)}(u+2\eta) &= \tau^{(1)}(u) + \delta(u) \\ \tau^{(1)}(u) \; \tau^{(0)}(u+4\eta) &= \tau^{(2)}(u) + \delta(u+2\eta)\tau^{(0)}(u) \\ \tau^{(2)}(u) \; \tau^{(0)}(u+6\eta) &= \tau^{(3)}(u) + \delta(u+4\eta)\tau^{(1)}(u) \\ &\vdots \\ \tau^{(n)}(u) \; \tau^{(0)}(u+[n+1]\cdot 2\eta) &= \tau^{(n+1)}(u) + \delta(u+n\cdot 2\eta)\tau^{(n-1)}(u) \; , \end{aligned} \tag{3.20}$$

where $\delta(u) \equiv \delta\{T(u)\}$ labels the PBC super quantum determinant (SQD) defined in appendix D. In contrast to ungraded models, such as the XXZ Heisenberg spin chain, this quantum determinant is *not* proportional to the identity.

IV. TQ-EQUATIONS FOR PBC

After applying a shift $u \to u - [n+1]\cdot 2\eta$ the PBC fusion hierarchy (3.20) reads

$$\tau^{(n)}(u-[n+1]\cdot 2\eta)\ \tau^{(0)}(u) = \tau^{(n+1)}(u-[n+1]\cdot 2\eta) + \delta(u-2\eta)\ \tau^{(n-1)}(u-[n+1]\cdot 2\eta)\ . \tag{4.1}$$

As all operators in this equation mutually commute, it may equally well be read as an equation for the eigenvalues $\Lambda^{(n)}(u)$ of the fused super transfer matrices. With $\Lambda(u) \equiv \Lambda^{(0)}(u)$ this yields

$$\Lambda(u) = \frac{\Lambda^{(n+1)}(u-[n+1]\cdot 2\eta)}{\Lambda^{(n)}(u-[n+1]\cdot 2\eta)} - (-1)^{N+M}\zeta^N(u)\frac{\Lambda^{(n-1)}(u-[n+1]\cdot 2\eta)}{\Lambda^{(n)}(u-[n+1]\cdot 2\eta)} \tag{4.2}$$

where M is the number of particles in the system, such that the sign $(-1)^M$ depends on the parity of the corresponding eigenstate (bosonic/fermionic). This pecularity stems from the fact, that the PBC SQD (D21a) can not simply be treated as a scalar function but rather as an operator that intersperses sign factors into the respective sectors. This may be illustrated by considering the fusion hierarchy (4.1) in a diagonal basis for chain length $N = 1$,

$$\begin{pmatrix} * & \\ & * \end{pmatrix}\begin{pmatrix} * & \\ & * \end{pmatrix} = \begin{pmatrix} * & \\ & * \end{pmatrix} + \begin{pmatrix} + & \\ & - \end{pmatrix}\begin{pmatrix} * & \\ & * \end{pmatrix} \quad \begin{matrix} \leftarrow B \\ \leftarrow F \end{matrix} \tag{4.3}$$

Introducing the functions

$$\bar{Q}^{(n)}(u) \equiv \Lambda^{(n)}(u-[n+1]\cdot 2\eta) \tag{4.4}$$

the eigenvalues can be rewritten as

$$\Lambda(u) = \frac{\bar{Q}^{(n+1)}(u+2\eta)}{\bar{Q}^{(n)}(u)} - (-1)^{N+M}\zeta^N(u)\frac{\bar{Q}^{(n-1)}(u-2\eta)}{\bar{Q}^{(n)}(u)}\ . \tag{4.5}$$

Now factorize $\bar{Q}^{(n)}$ according to

$$\bar{Q}^{(n)} = \chi_M(u)\cdot\Upsilon_n^N(u)\cdot Q^{(n)}(u) \tag{4.6}$$

where

$$\chi_M(u) \equiv \mathrm{e}^{\mathrm{i}\pi(M+1)\frac{u}{2\eta}} \quad \text{and} \quad \Upsilon_n(u) \equiv \prod_{k=0}^{n}\frac{\sin(u-[n-k+1]\cdot 2\eta)}{\sin(2\eta)}\ . \tag{4.7}$$

Provided the existence of the limit $Q(u) \equiv \lim_{n\to\infty} Q^{(n)}(u)$, this yields

$$\Lambda(u) = \left(\frac{\sin(u+2\eta)}{\sin(2\eta)}\right)^N \frac{Q(u-2\eta)}{Q(u)} - (-1)^M \left(\frac{\sin(u)}{\sin(2\eta)}\right)^N \frac{Q(u+2\eta)}{Q(u)} \ . \tag{4.8}$$

Due to the structure of the entries in the Lax-operators, the Q-functions factorize

$$Q(u) = \prod_{\ell=1}^{G} \sin(u-\lambda_\ell) \tag{4.9}$$

where the integer G can be determined by considering the asymptotic behaviour of $\Lambda(u)$. In the limit $z \equiv \mathrm{e}^{\mathrm{i}u} \to \infty$ the leading contribution to (4.8) is

$$\Lambda(u) \approx \mathrm{e}^{\mathrm{i}Nu} \left(\frac{\mathrm{e}^{\mathrm{i}\eta}}{\mathrm{e}^{\mathrm{i}\,2\eta} - \mathrm{e}^{-\mathrm{i}\,2\eta}}\right)^N \left[\mathrm{e}^{\mathrm{i}N\eta}\ \mathrm{e}^{-\mathrm{i}G\,2\eta} - (-1)^M \mathrm{e}^{-\mathrm{i}N\eta}\ \mathrm{e}^{\mathrm{i}G\,2\eta}\right] \tag{4.10}$$

such that consistency with (2.13) immediately fixes $G = M$. The requirement for the eigenvalues $\Lambda(u)$ to be analytic ultimately yields

$$\mathrm{Res}_{\lambda_j}(\Lambda) \stackrel{!}{=} 0 \quad \Leftrightarrow \quad \left(\frac{\sin(\lambda_j+2\eta)}{\sin(\lambda_j)}\right)^N \stackrel{!}{=} \prod_{\ell=1}^{M} \frac{\sin(\lambda_j-\lambda_\ell+2\eta)}{\sin(\lambda_\ell-\lambda_j+2\eta)} \tag{4.11}$$

which are precisely the Bethe equations for this model [4, 5, 13].

V. TRUNCATION OF THE PBC FUSION HIERARCHY

In the case of the XXZ-model it has been observed, that for certain values of the anisotropy η the fusion hierarchy repeats itself after a finite number of steps. The small polaron model shares this feature at values $\eta = \eta_p$ where

$$\eta_p \equiv \frac{\pi/2}{p+1} \ . \tag{5.1}$$

A. R-matrix truncation

The truncation identities for the R-matrices are found to be

$$\mathcal{R}_q^{(p)}(u,\eta_p) = \begin{pmatrix} -\mathcal{M}_p(u)\ \sigma_q^z & & \\ & \zeta(u)\sigma_q^z\ \mathcal{R}_q^{(p-2)}(u+2\eta_p,\eta_p) & \\ & & \mathcal{M}_p(u)\ (\sigma_q^z)^p \end{pmatrix} \tag{5.2}$$

where

$$\begin{aligned} \mathcal{R}_q^{(p)}(u,\eta) &\equiv B_{\ll 1\ldots(p+1)\gg}\ R_{\ll 1\ldots(p+1)\gg q}(u)\ B^{-1}_{\ll 1\ldots(p+1)\gg} \\ \mathcal{M}_p(u) &\equiv \left(\frac{1/2}{\sin(2\ \eta_p)}\right)^p \frac{\sin([p+1]\ u)}{\sin(2\ \eta_p)} \end{aligned} \tag{5.3}$$

with the transformation matrices $B_{\ll 1\ldots n\gg}$ explicitly given in appendix E up to $n = 4$.

B. Super transfer matrix truncation

For periodic boundary conditions the B-transformed fused monodromy matrix $\mathcal{T}^{(p)}(u,\eta)$ of an N-site model with quantum space $\mathcal{H} = V_{q_1} \otimes_s V_{q_2} \otimes_s \ldots \otimes_s V_{q_N}$ is defined as

$$\begin{aligned}
\mathcal{T}^{(p)}(u,\eta) &\equiv \mathcal{R}^{(p)}_{q_N}(u,\eta)\, \mathcal{R}^{(p)}_{q_{N-1}}(u,\eta)\ \ldots \mathcal{R}^{(p)}_{q_1}(u,\eta) \\
&= B_{\ll 1\ldots(p+1)\gg}\, R_{\ll 1\ldots(p+1)\gg q_N}(u)\ \ldots\ R_{\ll 1\ldots(p+1)\gg q_1}(u)\, B^{-1}_{\ll 1\ldots(p+1)\gg} \\
&= B_{\ll 1\ldots(p+1)\gg}\, T_{\ll 1\ldots(p+1)\gg}(u)\, B^{-1}_{\ll 1\ldots(p+1)\gg}
\end{aligned} \tag{5.4}$$

and due to the cyclic invariance of the supertrace it yields the exact same transfer matrix

$$\tau^{(p)}(u,\eta) \equiv \mathrm{str}_{\ll 1\ldots(p+1)\gg} \left\{ T_{\ll 1\ldots(p+1)\gg}(u) \right\} = \mathrm{str}_{\ll 1\ldots(p+1)\gg} \left\{ \mathcal{T}^{(p)}(u,\eta) \right\} \ . \tag{5.5}$$

At $\eta = \eta_p$ the truncation identity (5.2) for R-matrices gives

$$\mathcal{T}^{(p)}(u,\eta_p) = \begin{pmatrix} [-\mathcal{M}_p(u)]^N \prod_{i=N}^{1} \sigma^z_{q_i} & & \\ & \zeta^N(u) \prod_{i=N}^{1} \sigma^z_{q_i}\, \mathcal{R}^{(p-2)}_{q_i}(u+2\eta_p,\eta_p) & \\ & & [\mathcal{M}_p(u)]^N \prod_{i=N}^{1} (\sigma^z_{q_i})^p \end{pmatrix} \tag{5.6}$$

such that the truncation identity for the transfer matrices is found to be

$$\begin{aligned}
\tau^{(p)}(u,\eta_p) = [-\mathcal{M}_p(u)]^N \left(\prod_{i=1}^{N} \sigma^z_{q_i} \right) - (-1)^p [\mathcal{M}_p(u)]^N \left(\prod_{i=1}^{N} (\sigma^z_{q_i})^p \right) \\
- \zeta^N(u) \left(\prod_{i=1}^{N} \sigma^z_{q_i} \right) \tau^{(p-2)}(u+2\eta_p,\eta_p) \ .
\end{aligned} \tag{5.7}$$

VI. THE SMALL POLARON WITH OPEN BOUNDARY CONDITIONS

A. Reflection algebras and boundary matrices

The construction of integrable systems with open boundary conditions is based on representations of the graded reflection algebra

$$\begin{aligned}
&R_{12}(u-v)\mathcal{T}^-_1(u) R_{21}(u+v) \mathcal{T}^-_2(v) \\
&\qquad = \mathcal{T}^-_2(v) R_{12}(u+v) \mathcal{T}^-_1(u) R_{21}(u-v)
\end{aligned} \tag{6.1}$$

and the corresponding dual graded reflection algebra

$$\begin{aligned}
&\bar{R}_{12}(v-u)\mathcal{T}^+_1(u)^{\mathrm{st}_1} R_{21}(-u-v-4\eta) \mathcal{T}^+_2(v)^{\mathrm{ist}_2} \\
&\qquad = \mathcal{T}^+_2(v)^{\mathrm{ist}_2} R_{12}(-u-v-4\eta) \mathcal{T}^+_1(u)^{\mathrm{st}_1} \bar{R}_{21}(v-u).
\end{aligned} \tag{6.2}$$

The relation between $R_{ab}(u)$ and the *conjugated R-matrix* $\bar{R}_{ab}(u)$ is explained in appendix B. *c*-number valued boundary matrices, compatible with the respective reflection equation are found to be

$$
\begin{aligned}
K^-(u) &= \omega^- \begin{pmatrix} \sin(u+\psi_-) & \alpha_- \sin(2u) \\ \beta_- \sin(2u) & -\sin(u-\psi_-) \end{pmatrix} \\
K^+(u) &= \omega^+ \begin{pmatrix} \sin(u+2\eta+\psi_+) & \alpha_+ \sin(2[u+2\eta]) \\ \beta_+ \sin(2[u+2\eta]) & \sin(u+2\eta-\psi_+) \end{pmatrix}
\end{aligned}
\tag{6.3}
$$

with normalizations $\omega^\pm \equiv \omega^\pm(\eta)$ defined by

$$
\omega^-(\eta) \equiv \frac{1}{\sin(\psi_-)} \qquad \text{and} \qquad \omega^+(\eta) \equiv \frac{1}{2\cos(2\eta)\sin(\psi_+)} \,. \tag{6.4}
$$

These matrices share the periodicity property of the R-matrix, i.e.

$$
K^\mp(u+\pi) = -\sigma^z \; K^\mp(u) \; \sigma^z \,. \tag{6.5}
$$

Here the normalizations were chosen such that

$$
K^-(0) = \mathbb{1} \qquad \text{and} \qquad \mathrm{str}\,\{\, K^+(0) \,\} = 1 \,, \tag{6.6}
$$

but apart from this, the two solutions are related via

$$
K^+(u) = \left[\frac{1}{2\cos(2\eta)} K^-(-u-2\eta) \; \sigma^z \right]_{(\ominus \,\to\, \oplus)} \,, \tag{6.7}
$$

where $(\ominus \;\to\; \oplus)$ marks the replacements $(\alpha_-,\; \beta_-,\; \psi_-) \;\to\; (-\alpha_+,\; \beta_+,\; -\psi_+)$. In principle the parameters $\psi_\pm$ are arbitrary even Grassmann numbers but their invertability requires them to have a non-vanishing complex part[4]. The remaining parameters $\alpha_\pm$ and $\beta_\pm$ are odd Grassmann numbers, being subject to the condition $\alpha_\pm \cdot \beta_\pm = 0$.

Given the monodromy matrix $T(u) = L_N(u)L_{N-1}(u)\ldots L_1(u)$ it is possible to construct a further representation of the reflection algebra (6.1) as

$$
\mathcal{T}^-(u) = T(u) \; K^-(u) \; \widehat{T}(u) \equiv \begin{pmatrix} \mathcal{A}(u) & \mathcal{B}(u) \\ \mathcal{C}(u) & \mathcal{D}(u) \end{pmatrix} , \tag{6.8}
$$

---

[4] Such an additive part, that contains no nilpotent generators, is sometimes called the *body* of a Grassmann number. It is to be distinguished from the *soul* of a Grassmann number, which contains only sums of products of nilpotent generators.

with $\widehat{T}(u)$ being a shorthand notation for $T^{-1}(-u)$,

$$\begin{aligned}
\widehat{T}(u) &\equiv R_{01}^{-1}(-u)\ R_{02}^{-1}(-u)\ \ldots\ R_{0N}^{-1}(-u) \\
&\overset{(2.5d)}{=} \frac{1}{\zeta^N(u)} R_{10}(u)\ R_{20}(u)\ \ldots\ R_{N0}(u) \\
&\overset{(2.5a)}{=} \left(\frac{1}{\zeta(u)}\right)^N R_{01}(u)\ R_{02}(u)\ \ldots\ R_{0N}(u)\ ,
\end{aligned} \tag{6.9}$$

resulting in an OBC super transfer matrix

$$\tau(u) \equiv \mathrm{str}_0 \left\{ K^+(u) \mathcal{T}^-(u) \right\} \tag{6.10}$$

Expanding $\tau(u)$ around $u = 0$ one obtains a Hamiltonian featuring the same bulk part (2.2) as the corrsponding PBC Hamiltonian. Defining the shorthands $\mathcal{N}_\pm \equiv \frac{1}{2}\csc(2\eta)\csc(\psi_+)\sin(2\eta \pm \psi_+)$, the resulting OBC Hamiltonian

$$\begin{aligned}
H^{\mathrm{OBC}} = & \sum_{j=1}^{N-1} H_{j,j+1} + \frac{1}{2}\cot(\psi_-)\left[\bar{n}_1 - n_1\right] + \left[\mathcal{N}_+\ \bar{n}_N - \mathcal{N}_-\ n_N\right] \\
& + \csc(\psi_-)\left[\alpha_-\ c_1 - \beta_-\ c_1^\dagger\right] + \csc(\psi_+)\left[\alpha_+\ c_N - \beta_+\ c_N^\dagger\right]
\end{aligned} \tag{6.11}$$

is derived from the set of open boundary transfer matrices by

$$\partial_u\ \tau(u)|_{u=0} = 2\ H^{\mathrm{OBC}} + \mathrm{const.}\ . \tag{6.12}$$

In the case of diagonal boundaries, i.e. $\alpha_\pm = \beta_\pm = 0$, Bethe equations can be derived using the algebraic Bethe ansatz. This allows for the computation of the transfer matrix eigenvalues and eigenvectors (cf. appendix C resp. [13]).

B. Asymptotic behaviour of the OBC transfer matrix

Just as for the PBC case the highest order contributions to the (super) transfer matrix in the limit $z \equiv \mathrm{e}^{\mathrm{i}u} \to \infty$ will be considered. In addition to the asymptotic behaviour (2.11) of the Lax operators $L_j(u)$ it is found

$$L_j^{-1}(-u) = \frac{4\sin(2\eta)}{2\mathrm{i}\,z}\begin{pmatrix} n_j + \mathrm{e}^{2\mathrm{i}\eta}\ \bar{n}_j & 0 \\ 0 & \bar{n}_j - \mathrm{e}^{2\mathrm{i}\eta}\ n_j \end{pmatrix} + \mathcal{O}\left(\frac{1}{z^2}\right) \tag{6.13a}$$

$$K^-(u) = \frac{\omega^-}{2\mathrm{i}}\left[z^2\begin{pmatrix} 0 & \alpha_- \\ \beta_- & 0 \end{pmatrix} + z\begin{pmatrix} \mathrm{e}^{\mathrm{i}\psi_-} & 0 \\ 0 & -\mathrm{e}^{-\mathrm{i}\psi_-} \end{pmatrix} + \mathcal{O}\left(\frac{1}{z}\right)\right] \tag{6.13b}$$

$$K^+(u) = \frac{\omega^+\ \mathrm{e}^{2\mathrm{i}\eta}}{2\mathrm{i}}\left[z^2\mathrm{e}^{2\mathrm{i}\eta}\begin{pmatrix} 0 & \alpha_+ \\ \beta_+ & 0 \end{pmatrix} + z\begin{pmatrix} \mathrm{e}^{\mathrm{i}\psi_+} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\psi_+} \end{pmatrix} + \mathcal{O}\left(\frac{1}{z}\right)\right] \tag{6.13c}$$

which yield the asymptotics of the OBC transfer matrix (6.10) and of their eigenvalues $\Lambda_M(u)$ respectively

$$\begin{aligned}\tau(u) &= (-1)^N \,\frac{\omega^+\omega^-}{4}\, \mathrm{e}^{4\mathrm{i}\eta}\,(\beta_+\alpha_- - \alpha_+\beta_-)\,z^4 \prod_{j=1}^{N} (\bar{n}_j - \mathrm{e}^{2\mathrm{i}\eta} n_j)(n_j + \mathrm{e}^{2\mathrm{i}\eta}\bar{n}_j) + \mathcal{O}(z^2) \\ \Lambda_\pm(u) &= \pm(-1)^N \,\frac{\omega^+\omega^-}{4}\, \mathrm{e}^{4\mathrm{i}\eta}\,(\beta_+\alpha_- - \alpha_+\beta_-)\,z^4 \mathrm{e}^{\mathrm{i}N\,2\eta} + \mathcal{O}(z^2)\ .\end{aligned} \tag{6.14}$$

The eigenvalues $\Lambda_\pm(u)$ have been classified according to a parity which is determined by the (diagonal) operator controlling the asymptotics of $\tau(u)$.

Note that in the diagonal case, i.e. $\alpha_\pm = \beta_\pm = 0$, the $\mathcal{O}(z)$ terms of the K matrices become relevant such that

$$\begin{aligned}\tau(u) &= -(-1)^N \,\frac{\omega^+\omega^-}{4}\, \mathrm{e}^{2\mathrm{i}\eta}\,z^2 \Bigg[\mathrm{e}^{\mathrm{i}(\psi_+ + \psi_-)} \prod_{j=1}^{N} (n_j + \mathrm{e}^{2\mathrm{i}\eta}\bar{n}_j)(n_j + \mathrm{e}^{2\mathrm{i}\eta}\bar{n}_j) \\ &\qquad\qquad + \mathrm{e}^{-\mathrm{i}(\psi_+ + \psi_-)} \prod_{j=1}^{N} (\bar{n}_j - \mathrm{e}^{2\mathrm{i}\eta} n_j)(\bar{n}_j - \mathrm{e}^{2\mathrm{i}\eta} n_j)\Bigg] + \mathcal{O}(z) \\ \Lambda_M(u) &= -(-1)^N \,\frac{\omega^+\omega^-}{4}\, \mathrm{e}^{2\mathrm{i}\eta}\,z^2 \left(\mathrm{e}^{\mathrm{i}(\psi_+ + \psi_-)}\mathrm{e}^{4\mathrm{i}(N-M)\eta} + \mathrm{e}^{-\mathrm{i}(\psi_+ + \psi_-)}\mathrm{e}^{4\mathrm{i}M\eta}\right) + \mathcal{O}(z)\ .\end{aligned} \tag{6.15}$$

Here, as in the case of periodic boundary conditions, the asymptotic behaviour of the transfer matrix eigenvalues can be related to the (conserved) total number M of particles in the state.

C. Fusion of the boundary matrices

For the sake of readability it is convenient to define the following ordered product of R-matrices,

$$R_i^{\mathrm{string}}(u) \equiv \prod_{k=1}^{i} R_{k,i+1}(2u + [i+k-1]\cdot 2\eta)\ , \tag{6.16}$$

such that the fused K^- boundary matrices may be written as,

$$\begin{aligned}K^-_{(12\ldots n)}(u) &\equiv P^+_{12\ldots n} \left[\prod_{i=1}^{n-1} K_i^-(u + [i-1]\cdot 2\eta)\ R_i^{\mathrm{string}}(u)\right] K_n^-(u + [n-1]\cdot 2\eta)\ P^+_{12\ldots n} \\ \Rightarrow A_{(12\ldots n)}\ K^-_{(12\ldots n)}(u)\ A^{-1}_{(12\ldots n)} &\equiv \left(\begin{array}{c|c} K^-_{\ll 1\ldots n \gg}(u) & \\ \hline & \begin{matrix} 0 & \\ & \ddots \end{matrix} \end{array}\right)\ ,\end{aligned} \tag{6.17}$$

where $K^-_{(12\ldots n)}(u)$ is a $2^n \times 2^n$-matrix with $K^-_{\ll 1\ldots n \gg}(u)$ being the only non-vanishing block of dimensions $(n+1)\times(n+1)$. There is a useful relation between the fused K^-- and K^+-matrices

that stems from (6.7),

$$K^{+}_{(12\ldots n)}(u) \;=\; \left[\left(\frac{1}{2\cos(2\eta)}\right)^{n} K^{-}_{(n\ldots 21)}(-u-n\cdot 2\eta)\ \sigma^{z}_{(n)}\right]_{(\ominus\,\to\,\oplus)}$$

$$\Rightarrow\ A_{(12\ldots n)}\ K^{+}_{(12\ldots n)}(u)\ A^{-1}_{(12\ldots n)} \equiv \left(\begin{array}{c|cc} K^{+}_{\ll 1\ldots n\gg}(u) & & \\ \hline & 0 & \\ & & \ddots \end{array}\right) \tag{6.18}$$

and defines the $(n+1)\times(n+1)$-matrix $K^{+}_{\ll 1\ldots n\gg}(u)$ in the obvious way, where $\sigma^{z}_{(n)}$ was defined in (3.16). Note that the order of all spaces in $K^{-}_{(n\ldots 21)}$ is inverted. Thus, by changing the space labels according to $i\to n+1-i$ the fused right boundary matrix may explicitly be written as

$$K^{+}_{(12\ldots n)}(u) \;=\; P^{+}_{12\ldots n}\left[\prod_{i=1}^{n-1} K^{+}_{n+1-i}(u+[n-i]\cdot 2\eta)\ \bar{R}^{\mathrm{string}}_{i}(u)\right] K^{+}_{1}(u)\ P^{+}_{12\ldots n} \tag{6.19}$$

$$\bar{R}^{\mathrm{string}}_{i}(u) \;\equiv\; \prod_{k=1}^{i} \bar{R}_{n+1-k,n+1-(i+1)}(-2u+[i+k-1-2n]\cdot 2\eta)\ . \tag{6.20}$$

The reason why the conjugated R-matrices (B8) appear in this expression is that by commuting the σ^{z}-matrices, arising from (6.7), to the right, the relation

$$\bar{R}_{ab}(u)=\sigma^{z}_{a}R_{ab}(u)\sigma^{z}_{a}=\sigma^{z}_{b}R_{ab}(u)\sigma^{z}_{b} \tag{6.21}$$

is employed, cf. (B7).

Since $[P^{+}_{(1\ldots n)},\sigma^{z}_{(n)}]=0$ and $[\sigma^{z}_{(n)},A_{(12\ldots n)}\ \sigma^{z}_{(n)}\ A^{-1}_{(12\ldots n)}]=0$, the periodicity property (6.5) carries over to the fused K^{-}-matrices,

$$K^{\mp}_{(12\ldots n)}(u+\pi) \;=\; (-1)^{n}\ \sigma^{z}_{(n)}K^{\mp}_{(12\ldots n)}(u)\sigma^{z}_{(n)} \tag{6.22a}$$

$$K^{\mp}_{\ll 12\ldots n\gg}(u+\pi) \;=\; (-1)^{n}\ \sigma^{z}_{\ll n\gg}K^{\mp}_{\ll 12\ldots n\gg}(u)\sigma^{z}_{\ll n\gg} \tag{6.22b}$$

where the alternating sign results from successive application of (2.5f).

D. Fusion hierarchy for OBC

From the fused quantities, it is again possible to derive a family of commuting operators

$$\tau^{(n)}(u)\equiv \mathrm{str}_{\ll 1\ldots n\gg}\left\{K^{+}_{\ll 1\ldots n\gg}(u)\ T_{\ll 1\ldots n\gg}(u)\ K^{-}_{\ll 1\ldots n\gg}(u)\ \widehat{T}_{\ll 1\ldots n\gg}(u+[n-1]\cdot 2\eta)\right\} \tag{6.23}$$

that extends the existing familiy of commuting super transfer matrices $\tau(u)=\tau^{(1)}(u)$ such that $[\tau^{(i)}(u),\tau^{(k)}(v)]=0$ for all $i,j\geq 1$. The quantity $\widehat{T}_{\ll 1\ldots n\gg}(u)$ appearing in (6.23) is related to the

fused object

$$\widehat{T}_{(1\dots n)}(u+[n-1]\cdot 2\eta) \;=\; P^{+}_{1\dots n}\;\widehat{T}_1(u)\widehat{T}_2(u+2\eta)\cdot\ldots\cdot\widehat{T}_n(u+[n-1]\cdot 2\eta)\;P^{+}_{1\dots n}$$
$$= \prod_{i=1}^{N}\frac{R_{(1\dots n)q_i}(u,\eta)}{\zeta(u)\zeta(u+2\eta)\cdot\ldots\cdot\zeta(u+[n-1]\cdot 2\eta)}\;. \tag{6.24}$$

in the usual way by restriction to the only relevant matrixblock after applying the respective A-transformation. In the case of general open boundaries the fusion hierarchy for $n\geq 1$ is found to be

$$\tau^{(n)}(u)\cdot\tau^{(1)}(u+n\cdot 2\eta) = -\frac{\tau^{(n+1)}(u)}{\xi_n(u)} + \frac{\Delta(u+[n-1]\cdot 2\eta)}{\zeta(2u+2n\cdot 2\eta)}\cdot\xi_{n-1}(u)\tau^{(n-1)}(u) \tag{6.25}$$

where $\Delta(u)$ labels the OBC super quantum determinant defined in (D23) and

$$\xi_n(u)\equiv\prod_{k=1}^{n}\zeta(2u+[n+k]\cdot 2\eta)\;. \tag{6.26}$$

The structure of this fusion hierarchy can be further simplified by introducing the rescaled quantities

$$\tilde{\Delta}(u)\equiv\frac{\Delta(u)}{\zeta(2u+2\cdot 2\eta)} \qquad\text{and}\qquad \tilde{\tau}^{(n)}(u)\equiv-\left(\prod_{i=1}^{n-1}\xi_i^{-1}(u)\right)\tau^{(n)}(u) \tag{6.27}$$

with the convenient definitions $\tilde{\tau}^{(0)}(u)\equiv-\tau^{(0)}(u)\equiv\mathbb{1}$ and $\tilde{\tau}^{(1)}(u)\equiv-\tau^{(1)}(u)$ such that (6.25) becomes

$$\tilde{\tau}^{(n)}(u)\cdot\tilde{\tau}^{(1)}(u+n\cdot 2\eta) = \tilde{\tau}^{(n+1)}(u) - \tilde{\Delta}(u+[n-1]\cdot 2\eta)\cdot\tilde{\tau}^{(n-1)}(u)\;. \tag{6.28}$$

VII. TQ-EQUATIONS FOR OBC

As in the PBC case, the fusion hierarchy (6.28) provides a system of relations between the eigenvalues $\tilde{\Lambda}^{(n)}(u)$ of the fused (super) transfer matrices. Defining $\tilde{\Lambda}(u)\equiv\tilde{\Lambda}^{(1)}(u)$ and after shifting $u\to u-n\cdot 2\eta$ this yields

$$\tilde{\Lambda}(u) = \frac{\tilde{\Lambda}^{(n+1)}(u-n\cdot 2\eta)}{\tilde{\Lambda}^{(n)}(u-n\cdot 2\eta)} - \tilde{\Delta}(u-2\eta)\frac{\tilde{\Lambda}^{(n-1)}(u-n\cdot 2\eta)}{\tilde{\Lambda}^{(n)}(u-n\cdot 2\eta)} \tag{7.1}$$

Introducing the functions

$$h^{(n)}(u)\;\gamma^{(n)}(u)\;\tilde{Q}^{(n)}(u)\equiv\tilde{\Lambda}^{(n)}(u-n\cdot 2\eta) \tag{7.2}$$

where

$$\gamma^{(n)}(u) \equiv \frac{\sin(2u+2\eta)}{\sin(2u)} \prod_{j=1}^{n} \frac{\sin(2u-[2j-2]\cdot 2\eta)}{\sin(2u-[2j-3]\cdot 2\eta)} \tag{7.3a}$$

$$h^{(n)}(u) \equiv -(-1)^n \prod_{k=0}^{n} \omega^+ \sin(u-k\cdot 2\eta-\psi_+)\cdot \omega^- \sin(u-k\cdot 2\eta-\psi_-) \tag{7.3b}$$

the eigenvalues can be written as

$$\begin{aligned}\tilde{\Lambda}(u) = \mathfrak{K}^+_\delta(u)\mathfrak{K}^-_\delta(u+2\eta)\frac{\sin(2u)}{\sin(2u+2\eta)}\frac{\tilde{Q}^{(n+1)}(u+2\eta)}{\tilde{Q}^{(n)}(u)} \\ -\frac{\tilde{\Delta}(u-2\eta)}{\mathfrak{K}^+_\delta(u-2\eta)\mathfrak{K}^-_\delta(u)}\frac{\sin(2u-2\eta)}{\sin(2u-4\eta)}\frac{\tilde{Q}^{(n-1)}(u-2\eta)}{\tilde{Q}^{(n)}(u)}\end{aligned} \tag{7.4}$$

where the functions $\mathfrak{K}^\pm_\delta(u)$ are defined in (C7). Now assume that there exists a limit $\tilde{Q}(u) \equiv \lim_{n\to\infty}\tilde{Q}^{(n)}(u)$ which factorizes according to

$$\tilde{Q}(u) = f^N(u)\tilde{q}(u) \quad \text{with} \quad f(u) \equiv \mathrm{e}^{\mathrm{i}\pi\frac{u}{2\eta}}\frac{\sin(u-2\eta)}{\sin(u)} \ . \tag{7.5}$$

Resubstituting $\Lambda(u) = -\tilde{\Lambda}(u)$ by virtue of (6.27) it follows that

$$\Lambda(u) = H_\alpha(u)\,\frac{\tilde{q}(u-2\eta)}{\tilde{q}(u)} - H_\delta(u)\,\frac{\tilde{q}(u+2\eta)}{\tilde{q}(u)} \ . \tag{7.6}$$

where

$$\begin{aligned}H^\alpha(u) &\equiv \frac{\sin(2u+4\eta)}{\sin(2u+2\eta)}\,\mathfrak{K}^+_\alpha(u-2\eta)\mathfrak{K}^-_\alpha(u)\left(\frac{-\sin^2(u+2\eta)}{\sin(u+2\eta)\sin(u-2\eta)}\right)^N \\ H^\delta(u) &\equiv \frac{\sin(2u)}{\sin(2u+2\eta)}\,\mathfrak{K}^+_\delta(u)\mathfrak{K}^-_\delta(u+2\eta)\left(\frac{-\sin^2(u)}{\sin(u+2\eta)\sin(u-2\eta)}\right)^N\end{aligned} \tag{7.7}$$

such that $H^\alpha(u)H^\delta(u-2\eta) = \zeta^{-1}(2u)\Delta(u-2\eta)$. In the case $\tilde{q}(u) = q(u)$ this is precisely the result (C9) obtained by means of Bethe ansatz [13–15].

By matching the the asymptotically leading terms on the left and right hand side of their TQ-relations, the authors of [20] derived a set of constraints among the boundary parameters of the open (non-diagonal) XXZ Heisenberg spin chain which permitted them to employ Q-functions of the same factorized structure as in the case of diagonal boundary conditions, see Eq. (4.9). Unfortunately, this approach is not applicable to the small polaron model since in the non-diagonal case the leading term of $H^{\alpha,\delta}(u)$ does not have the same power in $z = \mathrm{e}^{\mathrm{i}u}$ as the leading term of $\Lambda(u)$. However, it is possible to construct Q-functions which at least yield the correct behaviour in the asymptotic regime. Based on observations for small system sizes we suggest that in the case of non-diagonal boundary conditions the Q-functions can be written as

$$\tilde{q}(u) = q(u) + \rho(u)\cdot(\beta_+\alpha_- - \alpha_+\beta_-) \tag{7.8}$$

with $q(u)$ being the Q-function allowing for factorization (C8) as in the diagonal case and another unknown function $\rho(u)$ depending on η and the diagonal boundary parameters $\psi_\pm$. To determine $\rho(u)$ the ansatz (7.8) should be used in the TQ-equation (7.6) together with the analytical properties of the transfer matrix eigenvalues, in particular their asymptotic behaviour (6.14).

VIII. TRUNCATION OF THE OBC FUSION HIERARCHY

From here on, for the sake of readability, some of the functions introduced above will be equipped with a second parameter indicating for them to be taken at that particular value of the anisotropy η. For instance, $K^\pm(u,\rho) \equiv K^\pm(u)|_{\eta\to\rho}$ and so on and so forth.

A. K-matrix truncation

It is convenient to define the following functions

$$\mu_n^\pm(u) \;\equiv\; \pm\delta\left\{K^\pm(\mp u - 2\eta_n, \eta_n)\right\} \frac{\sin(2\eta_n)}{\sin(2u - 2\cdot 2\eta_n)} \prod_{k=2}^{2n} \frac{\sin(2u + k\cdot 2\eta_n)}{\sin(2\eta_n)} \tag{8.1}$$

$$\nu_n^\pm(u) \;\equiv\; \mp\frac{\omega_n^\pm}{\mu_n^\pm(u)} \left(\frac{\omega_n^\pm}{2}\right)^n \sin([n+1][u\mp\psi_\pm]) \prod_{i=1}^{n}\prod_{j=1}^{i} \frac{\sin(2u + [i+j]\cdot 2\eta_n)}{\sin(2\eta_n)} \;. \tag{8.2}$$

where $\omega_n^\pm \equiv \omega^\pm(\eta_n)$ and to introduce the shorthand notations

$$\begin{aligned} \mathcal{K}^-_{\ll n\gg}(u,\eta) &\equiv \sigma^z_{\ll n\gg}\cdot K^-_{\ll 1\ldots n\gg}(u+2\eta) \\ \mathcal{K}^+_{\ll n\gg}(u,\eta) &\equiv K^+_{\ll 1\ldots n\gg}(u+2\eta)\cdot\sigma^z_{\ll n\gg}\;. \end{aligned} \tag{8.3}$$

The truncation identities for the boundary matrices can then be expressed as

$$\begin{aligned} &C_{\ll 1\ldots n\gg}\, K^\pm_{\ll 1\ldots n\gg}(u,\eta_{n-1})\, C^{-1}_{\ll 1\ldots n\gg} \\ &= \mu^\pm_{n-1}(u) \begin{pmatrix} \nu^\pm_{n-1}(\mp u) & & \\ & B_{\ll 1\ldots n-2\gg}\mathcal{K}^\pm_{\ll n-2\gg}(u,\eta_{n-1})B^{-1}_{\ll 1\ldots n-2\gg} & * \\ & & (\pm 1)^n\nu^\pm_{n-1}(\pm u) \end{pmatrix} \end{aligned} \tag{8.4}$$

B. OBC super transfer matrix truncation

In order to be compatible with the truncation identities for the boundary matrices, the R-matrix truncation identities (5.2) need to be recast, this time employing the C transformation matrices,

$$C_{\ll 1\ldots n\gg}\ R_{\ll 1\ldots n\gg q}(u,\eta_{n-1})\ C^{-1}_{\ll 1\ldots n\gg} = \begin{pmatrix} -\mathcal{M}_{n-1}(u)\ \sigma^z_q & & \\ & \zeta(u)\ \sigma^z_q\ \mathcal{R}^{(n-2)}_q(u+2\eta_{n-1},\eta_{n-1}) & * \\ & & \mathcal{M}_{n-1}(u)\ (\sigma^z_q)^{n-1} \end{pmatrix} \tag{8.5}$$

where in slight contrast to definition (5.3)

$$\mathcal{R}^{(n)}_q(u,\eta) \equiv B_{\ll 1\ldots n\gg}\ R_{\ll 1\ldots n\gg q}(u)\ B^{-1}_{\ll 1\ldots n\gg} \tag{8.6}$$

such that for the single row monodromy matrix

$$\mathcal{T}^{(n)}(u,\eta)\ \equiv\ C_{\ll 1\ldots n\gg}\ R_{\ll 1\ldots n\gg q_N}(u)\cdot\ldots\cdot R_{\ll 1\ldots n\gg q_2}(u)R_{\ll 1\ldots n\gg q_1}(u)\ C^{-1}_{\ll 1\ldots n\gg} \tag{8.7}$$

$$\equiv\ C_{\ll 1\ldots n\gg}\ T_{\ll 1\ldots n\gg}(u)\ C^{-1}_{\ll 1\ldots n\gg} \tag{8.8}$$

the truncation identity at $\eta=\eta_{n-1}$ reads

$$\mathcal{T}^{(n)}(u,\eta_{n-1}) = \begin{pmatrix} [-\mathcal{M}_{n-1}(u)]^N \prod_{i=N}^1 \sigma^z_{q_i} & & \\ & \zeta^N(u) \prod_{i=N}^1 \sigma^z_{q_i}\ \mathcal{R}^{(n-2)}_{q_i}(u+2\eta_{n-1},\eta_{n-1}) & * \\ & & [\mathcal{M}_{n-1}(u)]^N \prod_{i=N}^1 (\sigma^z_{q_i})^{n-1} \end{pmatrix} \tag{8.9}$$

Again it is convenient to introduce the C-transformed object

$$\widehat{\mathcal{T}}^{(n)}(u,\eta) \equiv C_{\ll 1\ldots n\gg}\ \widehat{T}_{\ll 1\ldots n\gg}(u)\ C^{-1}_{\ll 1\ldots n\gg} \tag{8.10}$$

to easily recognize the truncation identity

$$\widehat{\mathcal{T}}^{(n)}(u+[n-1]\cdot 2\eta_{n-1},\eta_{n-1}) = \frac{1}{\zeta^N(u)\ \zeta^N(u+2\eta_{n-1})\ldots\zeta^N(u+[n-1]\cdot 2\eta_{n-1})} \times \tag{8.11}$$

$$\times \begin{pmatrix} [-\mathcal{M}_{n-1}(u)]^N \prod_{i=1}^N \sigma^z_{q_i} & & \\ & \zeta^N(u) \prod_{i=1}^N \sigma^z_{q_i}\ \mathcal{R}^{(n-2)}_{q_i}(u+2\eta_{n-1},\eta_{n-1}) & * \\ & & [\mathcal{M}_{n-1}(u)]^N \prod_{i=1}^N (\sigma^z_{q_i})^{n-1} \end{pmatrix}.$$

Now that the individual truncation identities for all the objects involved in the construction of the fused OBC super transfer matrix $\tau^{(n)}(u)$ are known, it can be shown by simple matrix multiplication[5] of (8.4$^+$), (8.9), (8.4$^-$) and (8.11) that

$$\tau^{(n)}(u,\eta_{n-1}) = \mathrm{str}_{\ll 1\ldots n\gg}\left\{\begin{pmatrix} X^+ & & \\ & Y & * \\ & & X^- \end{pmatrix}\right\} = X^+ - \mathrm{str}_{\ll 1\ldots n-2\gg}\{Y\} + (-1)^n\; X^- \tag{8.12}$$

with the placeholders $X^\pm$ and Y defined by

$$X^\pm \equiv (\pm 1)^n \;\left[\prod_{k=0}^{n-1}\zeta^{-N}(u+k\cdot 2\eta_{n-1})\right]\mathcal{M}^{2N}_{n-1}(u)\;\mu^+_{n-1}(u)\mu^-_{n-1}(u)\;\nu^+_{n-1}(\mp u)\nu^-_{n-1}(\pm u) \tag{8.13}$$

and

$$\begin{aligned} Y =\;& \phi^\tau_{n-1}(u)\;B_{\ll 1\ldots n-2\gg}\;K^+_{\ll 1\ldots n-2\gg}(u+2\eta_{n-1})\;\sigma^z_{\ll n-2\gg}\left(\prod_{i=1}^N\sigma^z_{q_i}\right)\times \\ &\times T_{\ll 1\ldots n-2\gg}(u+2\eta_{n-1})\;\sigma^z_{\ll n-2\gg}\;K^-_{\ll 1\ldots n-2\gg}(u+2\eta_{n-1})\times \\ &\times\left(\prod_{i=1}^N\sigma^z_{q_i}\right)\widehat{T}_{\ll 1\ldots n-2\gg}(u+2\eta_{n-1}+[(n-2)-1]\cdot 2\eta_{n-1})\;B^{-1}_{\ll 1\ldots n-2\gg} \\ =\;& \phi^\tau_{n-1}(u)\;B_{\ll 1\ldots n-2\gg}\;K^+_{\ll 1\ldots n-2\gg}(u+2\eta_{n-1})\;T_{\ll 1\ldots n-2\gg}(u+2\eta_{n-1})\times \\ &\times K^-_{\ll 1\ldots n-2\gg}(u+2\eta_{n-1})\;\widehat{T}_{\ll 1\ldots n-2\gg}(u+2\eta_{n-1}+[(n-2)-1]\cdot 2\eta_{n-1})\;B^{-1}_{\ll 1\ldots n-2\gg}\;. \end{aligned} \tag{8.14}$$

In the second step of equation (8.14) relation (D22) has been employed to get rid of the σ^z factors such that (8.12) eventually yields the truncation identities for the OBC transfer matrices,

$$\tau^{(n)}(u,\eta_{n-1}) = \phi^{\mathrm{id}}_{n-1}(u)\cdot\mathbb{1} - \phi^\tau_{n-1}(u)\cdot\tau^{(n-2)}(u+2\eta_{n-1},\eta_{n-1}) \tag{8.15}$$

where $\phi^{\mathrm{id}}_n(u)$ and $\phi^\tau_n(u)$ are rather lengthy expressions given by

$$\begin{aligned} \phi^{\mathrm{id}}_n(u) &= \left[\prod_{k=0}^{n}\zeta^{-N}(u+k\cdot 2\eta_n)\right]\mathcal{M}^{2N}_n(u)\;\mu^+_n(u)\mu^-_n(u)\;[\nu^+_n(-u)\nu^-_n(u)+\nu^+_n(u)\nu^-_n(-u)] \\ \phi^\tau_n(u) &= \left(\frac{\zeta(u)}{\zeta(u+n\cdot 2\eta_n)}\right)^N\;\mu^+_n(u)\mu^-_n(u)\;. \end{aligned} \tag{8.16}$$

In terms of the rescaled transfer matrices (6.28) it is reasonable to introduce

$$\begin{aligned} \tilde{\phi}^{\mathrm{id}}_n(u) &= -\left[\prod_{i=1}^n\xi_i^{-1}(u)\right]_{\eta=\eta_n}\phi^{\mathrm{id}}_n(u) \\ \tilde{\phi}^\tau_n(u) &= \left[\prod_{i=1}^n\xi_i^{-1}(u)\right]_{\eta=\eta_n}\phi^\tau_n(u)\left[\prod_{i=1}^{n-2}\xi_i(u+2\eta_n)\right]_{\eta=\eta_n}\;. \end{aligned} \tag{8.17}$$

[5] Due to the cyclic invariance of the supertrace, all the matrix objects in (6.23) may be conjugated by means of the C-transformation without changing the actual super transfer matrix.

which yield the respective rescaled truncation identities

$$\tilde{\tau}^{(n)}(u,\eta_{n-1}) = \tilde{\phi}^{\mathrm{id}}_{n-1}(u) \cdot \mathbb{1} - \tilde{\phi}^{\tau}_{n-1}(u) \cdot \tilde{\tau}^{(n-2)}(u+2\eta_{n-1},\eta_{n-1}) \ . \tag{8.18}$$

IX. SUMMARY AND CONCLUSION

Starting from structures provided by the Yang-Baxter algebra (2.6) and the reflection algebra (6.1), (6.2) we have set up the fusion hierarchies for the commuting transfer matrices $\tau^{(n)}(u)$ of the small polaron model with periodic and general open boundary conditions, respectively. Taking the limit $n \to \infty$ we have obtained TQ-equations for the eigenvalues of the transfer matrices which can be solved by functional Bethe ansatz methods for periodic and diagonal open boundary conditions. The resulting spectrum coincides with what has been found previously using the algebraic Bethe ansatz [4, 5, 13–15].

For generic non-diagonal boundary conditions the $U(1)$ symmetry of the model corresponding to particle number conservation is broken. Therefore, the algebraic approach cannot be applied as it uses the Fock vacuum as a reference state and relies on this being an eigenstate of the system. This situation is well known from the (ungraded) XXZ Heisenberg spin-1/2 chain with non-diagonal boundary fields where in spite of significant activities a practical solution of the eigenvalue problem for generic anisotropies and boundary fields is lacking. Here we have used the strategies employed for the XXZ chain to the (graded) small polaron chain: as for the spin chain the fusion hierarchy can be truncated at a finite order for anisotropies being roots of unity, $\eta = \pi/(2(p+1))$ [17]. We have derived the corresponding truncation identities for all boundary conditions considered. For generic anisotropies the spectral problem needs to be studied within its formulation as a TQ-equation.

To actually compute eigenvalues of the transfer matrices further steps have to be taken: for anisotropies being roots of unity the analysis of the truncated fusion hierarchy can be be done following the steps which have been established for the XXZ chain [27–29] where additional constraints on the boundary fields may arise. For generic anisotropies the situation is more complicated: in the ungraded XXZ chain a (factorized) Bethe ansatz for the Q-function given in terms of finitely many parameters was possible only if the boundary parameters satisfy a constraint [18–20, 23]. For graded models such a constraint may be absent: in the rational limit $\eta \to 0$ of the model considered here the functional form of the Q-function remained unchanged when off-diagonal boundary fields where added [26]. Similarly, the nilpotency of the off-diagonal boundary fields may allow for a general solution of the small polaron model. Starting with the proposed ansatz (7.8) for the Q-function the derivation of Bethe type equations appears to be possible in the generic case. These

open questions shall be addressed in a future publication.

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Appendix A: Graded vector spaces

Fermionic lattice models exhibit a natural $\mathbb{Z}_2$ gradation on their local space of states, i.e. $V = V_0 \oplus V_1$ is equipped with a notion of parity,

$$\mathrm{p} : V_i \to \mathbb{Z}_2 \quad , \quad \mathrm{p}(v_i) \mapsto i \in \{0, 1\} \ . \tag{A1}$$

Let $\dim V_0 \equiv m \in \mathbb{N}$ and $\dim V_1 \equiv n \in \mathbb{N}$ be finite. Then V is said to have dimension $(m|n)$ and V_0, V_1 are called the *homogeneous subspaces* of V. An element $v \in V$ is said to be *even* if $\mathrm{p}(v) = 0$ and is respectively called *odd* if $\mathrm{p}(v) = 1$. While even elements of V correspond to bosonic states, odd elements represent fermionic states. For instance consider the case where both of the homogeneous subspaces V_0 and V_1 are one-dimensional such that the composite local space of states $V = V_0 \oplus V_1$ is spanned by just one bosonic and one fermionic state. Then V is said to have BF-grading, where BF refers to an ordered basis of V in which the first basis vector is associated to with the bosonic state (B) whereas the second basis vector is associated to the fermionic state (F). Now consider the tensor product of two copies of V. Taking into account the order of the basis states, the tensor product space will have $BFFB$-grading,

$$V \otimes V = (V_0 \oplus V_1) \otimes (V_0 \oplus V_1) = \underbrace{(V_0 \otimes V_0)}_{B} \oplus \underbrace{(V_0 \otimes V_1)}_{F} \oplus \underbrace{(V_1 \otimes V_0)}_{F} \oplus \underbrace{(V_1 \otimes V_1)}_{B} \ . \tag{A2}$$

In the following, the conventions from [30] will essentially be adopted.

Let $\{e_1, e_2, \ldots, e_m, e_{m+1}, \ldots, e_{m+n}\}$ be a homogeneous basis of V, i.e. each basis element has distinct parity $\mathrm{p}(e_\alpha)$, and for convenience let this basis be ordered, such that the first m elements span the even and the last n elements span the odd subspace of V,

$$\mathrm{p}(\alpha) \equiv \mathrm{p}(e_\alpha) = \begin{cases} 0 & \text{if} \quad 1 \le \alpha \le m \\ 1 & \text{if} \quad m+1 \le \alpha \le m+n \end{cases} \ . \tag{A3}$$

In order to deal with an algebra of linear operators, acting on the graded local space of states, it is necessary to extend the concept of parity to $\mathrm{End}(V)$, the space of endomorphisms of V. The

$(m+n)\times(m+n)$ basis elements of $\mathrm{End}(V)$ will be labeled $e_\alpha^{\ \beta}$ and are defined through their action on the above basis of V,

$$e_\alpha^{\ \beta} e_\gamma \equiv \delta_\gamma^\beta e_\alpha \ . \tag{A4}$$

By extending the definition of the parity function to

$$\mathrm{p}(e_\alpha^{\ \beta}) \equiv \mathrm{p}(\alpha) + \mathrm{p}(\beta) \mod 2 \tag{A5}$$

$\mathrm{End}(V)$ becomes a $\mathbb{Z}_2$ graded vector space. A basis of the N-fold product space

$$\mathrm{End}^{\otimes N}(V) \equiv \underbrace{\mathrm{End}(V)\otimes\mathrm{End}(V)\otimes\ldots\otimes\mathrm{End}(V)}_{N \text{ times}} \tag{A6}$$

can most naturally be obtained by embedding the local basis elements $e_\alpha^{\ \beta}$ into this tensorproduct structure. Moreover, $\mathrm{End}^{\otimes N}(V)$ aquires a $\mathbb{Z}_2$ grading by a further extension of the definition of the parity function,

$$\mathrm{p}(e_{\alpha_1}^{\beta_1}\otimes e_{\alpha_2}^{\beta_2}\otimes\ldots\otimes e_{\alpha_N}^{\beta_N}) \equiv \mathrm{p}(e_{\alpha_1}^{\beta_1}) + \mathrm{p}(e_{\alpha_2}^{\beta_2}) + \ldots + \mathrm{p}(e_{\alpha_N}^{\beta_N}) \mod 2 \ . \tag{A7}$$

When dealing with graded vector spaces, it is useful to replace the usual tensorproduct structure by a so-called *super tensorproduct*. The symbol $\otimes_s$ will be used to distinguish this new structure. With respect to a certain basis, the components of the super tensorproruct of two operators $A \in \mathrm{End}^{\otimes k}(V)$ and $B \in \mathrm{End}^{\otimes l}(V)$, where $k, l \in \mathbb{N}$, is explicitly defined through

$$(A\otimes_s B)^{\alpha\gamma}_{\beta\delta} = (-1)^{[\mathrm{p}(\alpha)+\mathrm{p}(\beta)]\,\mathrm{p}(\gamma)} A^\alpha_{\ \beta} B^\gamma_{\ \delta} \ . \tag{A8}$$

As pointed out in [30], the super tensorproduct allows for a most convenient *graded* embedding of the $e_\alpha^{\ \beta}$ into the j-th subspace of $\mathrm{End}^{\otimes N}(V)$,

$$e_{j,\alpha}^{\ \ \ \beta} \equiv \mathbb{1}^{\otimes_s(j-1)} \otimes_s e_\alpha^{\ \beta} \otimes_s \mathbb{1}^{\otimes_s(N-j)} \ . \tag{A9}$$

A graded version of the permutation operator $\mathcal{P}$ is defined by the relation

$$\mathcal{P}(A\otimes_s B) = (B\otimes_s A)\mathcal{P} \ . \tag{A10}$$

If (A9) is employed as a basis for $\mathrm{End}^{\otimes N}(V)$, the operator $\mathcal{P}_{ij}$ which permutes the i-th and the j-th subspace can explicitly be constructed as

$$\mathcal{P}_{ij} = (-1)^{\mathrm{p}(\beta)} e_{i,\alpha}^{\ \ \ \beta}\, e_{j,\beta}^{\ \ \ \alpha} \ . \tag{A11}$$

In the following, the definitions of some well-known operations, namely the matrix transposition and the trace operation, will be adapted to fit the needs of graded vector spaces. A nicely motivated and much more elaborated list of matrix operations on graded vector spaces can found in [16].

- Firstly, the *super transposition* of an element $A \in \mathrm{End}(V)$ is defined by

$$(A^{\mathrm{st}})^{\alpha}_{\ \beta} = (-1)^{\mathrm{p}(\alpha)[\mathrm{p}(\alpha+\beta)]} A_{\beta}^{\ \alpha} \ . \tag{A12}$$

In contrast to the ungraded case, the super transposition is not an involution, i.e. applying the super transposition twice does not yield the identity operation. As pointed out in [8], it is therefore convenient to introduce an *inverse super transposition*,

$$(A^{\mathrm{ist}})^{\alpha}_{\ \beta} = (-1)^{\mathrm{p}(\beta)[\mathrm{p}(\alpha+\beta)]} A_{\beta}^{\ \alpha} \ . \tag{A13}$$

The *partial* super transposition, i.e. a super transposition on the j-th subspace of $\mathrm{End}^{\otimes N}(V)$, is defined through

$$(A_1 \otimes_s \ldots \otimes_s A_j \otimes_s \ldots \otimes_s A_N)^{\mathrm{st}_j} \equiv A_1 \otimes_s \ldots \otimes_s (A_j)^{\mathrm{st}} \otimes_s \ldots \otimes_s A_N \ . \tag{A14}$$

The partial inverse super transposition is defined analogously. Please note that, as opposed to ordinary partial matrix transpositions on ungraded vector spaces, the successive application of partial super transpositions on all subspaces is gernerally not equal to a total super transposition, i.e. $(A_1 \otimes_s A_2)^{\mathrm{st}_1\mathrm{st}_2} \neq (A_1 \otimes_s A_2)^{\mathrm{st}}$.

- Secondly, the *super trace* of some $A \in \mathrm{End}(V)$ is given by

$$\mathrm{str}\,\{\, A \,\} \equiv \sum_{\alpha} (-1)^{\mathrm{p}(\alpha)} A^{\alpha}_{\ \alpha} \ . \tag{A15}$$

For operators $B \in \mathrm{End}^{\otimes N}(V)$ it is convenient to define a *partial super trace* on subspace j as

$$\mathrm{str}_j \,\{\, B \,\}^{\alpha_1 \ldots \alpha_{j-1}\, \alpha_{j+1} \ldots \alpha_N}_{\beta_1 \ldots \beta_{j-1}\, \beta_{j+1} \ldots \beta_N} \equiv \sum_{\gamma} (-1)^{\mathrm{p}(\gamma)} B^{\alpha_1 \ldots \alpha_{j-1}\, \gamma\, \alpha_{j+1} \ldots \alpha_N}_{\beta_1 \ldots \beta_{j-1}\, \gamma\, \beta_{j+1} \ldots \beta_N} \ . \tag{A16}$$

Appendix B: Relation to Bracken's dual reflection algebra

According to [8] the dual reflection equation for quite general graded models reads

$$\begin{aligned} &R_{12}(v-u) K_1^{+}(u) \widetilde{\widetilde{R}}_{21}(-u-v)^{\mathrm{ist}_1\mathrm{st}_2} K_2^{+}(v) \\ &\qquad = K_2^{+}(v) \widetilde{R}_{12}(-u-v)^{\mathrm{ist}_1\mathrm{st}_2} K_1^{+}(u) R_{21}(v-u) \end{aligned} \tag{B1}$$

where

$$\widetilde{\widetilde{R}}_{21}(\lambda)^{\mathrm{ist}_1\mathrm{st}_2} \ = \ \left(\left[\left\{ R_{21}^{-1}(\lambda) \right\}^{\mathrm{ist}_2} \right]^{-1} \right)^{\mathrm{st}_2} \tag{B2}$$

$$\widetilde{R}_{12}(\lambda)^{\mathrm{ist}_1\mathrm{st}_2} \ = \ \left(\left[\left\{ R_{12}^{-1}(\lambda) \right\}^{\mathrm{st}_1} \right]^{-1} \right)^{\mathrm{ist}_1} \ . \tag{B3}$$

By performing a super transposition on the first space and an inverse super transposition on the second, i.e. by applying $(.)^{\mathrm{st}_1\mathrm{ist}_2}$ to equation (B1) one obtains the equivalent form

$$\begin{aligned} R_{21}(v-u)^{\mathrm{st}_1\mathrm{ist}_2} K_1^+(u)^{\mathrm{st}_1} \widetilde{R}_{12}(-u-v) K_2^+(v)^{\mathrm{ist}_2} \\ = K_2^+(v)^{\mathrm{ist}_2} \widetilde{\widetilde{R}}_{21}(-u-v) K_1^+(u)^{\mathrm{st}_1} R_{12}(v-u)^{\mathrm{st}_1\mathrm{ist}_2} \end{aligned} \tag{B4}$$

In the case of the small polaron R-matrix as defined in (2.3) one finds

$$\widetilde{\widetilde{R}}_{21}(\lambda) \;=\; \frac{\zeta(\lambda)}{\zeta(\lambda-2\eta)}\; R_{12}(\lambda-4\eta) \tag{B5}$$

$$\widetilde{R}_{12}(\lambda) \;=\; \frac{\zeta(\lambda)}{\zeta(\lambda-2\eta)}\; R_{21}(\lambda-4\eta) \tag{B6}$$

At this point it is convenient to introduce a shorthand, which will henceforth be referred to as *conjugated R-matrix*,

$$\bar{R}_{ba}(\lambda) \equiv M_a^{-1}\; R_{ba}(\lambda)\; M_a \tag{B7}$$

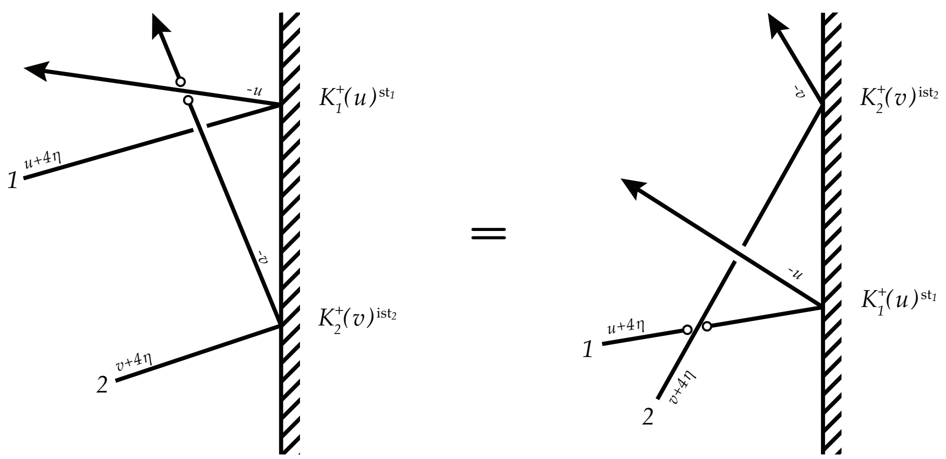
with M being the so-called crossing matrix. For the small polaron model in particular, it is found that $M = \sigma^z$ such that

$$\begin{aligned} \bar{R}_{ab}(\lambda) = R_{ba}^{\mathrm{st}_a\mathrm{ist}_b}(\lambda) &\overset{(2.5a)}{=} R_{ba}^{\mathrm{ist}_a\mathrm{st}_b}(\lambda) \\ = R_{ab}^{\mathrm{st}_a^2}(\lambda) = R_{ab}^{\mathrm{ist}_a^2}(\lambda) &= R_{ab}^{\mathrm{st}_b^2}(\lambda) = R_{ab}^{\mathrm{ist}_b^2}(\lambda)\ . \end{aligned} \tag{B8}$$

Using this conjugated R-matrix (B8), the dual reflection equation may be written as

$$\begin{aligned} \bar{R}_{12}(v-u) K_1^+(u)^{\mathrm{st}_1} R_{21}(-u-v-4\eta) K_2^+(v)^{\mathrm{ist}_2} \\ = K_2^+(v)^{\mathrm{ist}_2} R_{12}(-u-v-4\eta) K_1^+(u)^{\mathrm{st}_1} \bar{R}_{21}(v-u) \end{aligned} \tag{B9}$$

and is graphically depicted by



Appendix C: Algebraic Bethe ansatz for diagonal boundaries

The reflection equation (6.1) gives 16 fundamental commutation relations for the quantum space operators $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ of which the following three are of particular interest,

$$\mathcal{B}(u)\mathcal{B}(v) = \mathcal{B}(v)\mathcal{B}(u) \tag{C1}$$

$$\begin{aligned}\mathcal{A}(u)\mathcal{B}(v) =& \frac{\mathrm{s}_0(u+v)\,\mathrm{s}_2(v-u)}{\mathrm{s}_0(v-u)\,\mathrm{s}_2(u+v)}\mathcal{B}(v)\mathcal{A}(u)\\ &+ \frac{\vartheta(v)\,\mathrm{s}_2(0)}{\mathrm{s}_2(u+v)}\mathcal{B}(u)\left\{\frac{\mathrm{s}_0(2v)\,\mathrm{s}_2(u+v)}{\vartheta(v)\,\mathrm{s}_0(u-v)\,\mathrm{s}_2(2v)}\mathcal{A}(v) - \widetilde{\mathcal{D}}(v)\right\}\end{aligned} \tag{C2}$$

$$\begin{aligned}\widetilde{\mathcal{D}}(u)\mathcal{B}(v) =& \frac{\mathrm{s}_4(u+v)\,\mathrm{s}_2(u-v)}{\mathrm{s}_0(u-v)\,\mathrm{s}_2(u+v)}\mathcal{B}(v)\widetilde{\mathcal{D}}(u) - \frac{\mathrm{s}_2(0)\,\mathrm{s}_4(2u)\,\mathrm{s}_0(2v)}{\vartheta(u)\,\mathrm{s}_2(2u)\,\mathrm{s}_2(u+v)\,\mathrm{s}_2(2v)}\mathcal{B}(u)\times\\ &\times\left\{\frac{\vartheta(v)\,\mathrm{s}_2(2v)\,\mathrm{s}_2(u+v)}{\mathrm{s}_0(2v)\,\mathrm{s}_0(u-v)}\widetilde{\mathcal{D}}(v) - \mathcal{A}(v)\right\}\ ,\end{aligned} \tag{C3}$$

using the abbreviation $\mathrm{s}_k(\lambda) \equiv \sin(\lambda + k\eta)$. To obtain the desired commutation relations, it is necessary to make an ansatz for a shifted $\mathcal{D}$-operator

$$\mathcal{D}(\lambda) = \vartheta(\lambda)\widetilde{\mathcal{D}}(\lambda) + \phi(\lambda)\mathcal{A}(\lambda) \tag{C4}$$

and to determine the scalar functions $\phi(\lambda)$ and $\vartheta(\lambda)$. It turns out that $\phi(\lambda) = \frac{\mathrm{s}_2(0)}{\mathrm{s}_2(2\lambda)}$ while $\vartheta(\lambda)$ remains arbitrary. Starting from the general boundary matrices given in (6.3) the diagonal case can easily be obtained by setting $\alpha_\pm = \beta_\pm = 0$. This leads to Bethe equations

$$\left(\frac{\mathrm{s}_2(\nu_j)}{\mathrm{s}_0(\nu_j)}\right)^{2N} = \frac{\mathrm{s}_2(\nu_j-\psi_+)\,\mathrm{s}_2(\nu_j-\psi_-)}{\mathrm{s}_0(\nu_j+\psi_+)\,\mathrm{s}_0(\nu_j+\psi_-)}\prod_{\substack{\ell=1\\ \ell\neq j}}^{M}\frac{\mathrm{s}_4(\nu_j+\nu_\ell)\,\mathrm{s}_2(\nu_j-\nu_\ell)}{\mathrm{s}_0(\nu_j+\nu_\ell)\,\mathrm{s}_{-2}(\nu_j-\nu_\ell)} \tag{C5}$$

and super transfer matrix eigenvalues[6]

$$\begin{aligned}\Lambda(u) =& \mathfrak{K}_\alpha^-(u)\left(\mathfrak{K}_\alpha^+(u) - \frac{\mathrm{s}_2(0)}{\mathrm{s}_2(2u)}\mathfrak{K}_\delta^+(u)\right)\left(\frac{\mathrm{s}_2(u)}{\mathrm{s}_2(-u)}\right)^N\prod_{\ell=1}^{M}\frac{\mathrm{s}_0(u+\nu_\ell)\,\mathrm{s}_2(\nu_\ell-u)}{\mathrm{s}_0(\nu_\ell-u)\,\mathrm{s}_2(u+\nu_\ell)}\\ &-\mathfrak{K}_\delta^+(u)\left(\mathfrak{K}_\delta^-(u) - \frac{\mathrm{s}_2(0)}{\mathrm{s}_2(2u)}\mathfrak{K}_\alpha^-(u)\right)\left(\frac{\mathrm{s}_0^2(u)}{\mathrm{s}_2(u)\,\mathrm{s}_2(-u)}\right)^N\prod_{\ell=1}^{M}\frac{\mathrm{s}_4(u+\nu_\ell)\,\mathrm{s}_2(u-\nu_\ell)}{\mathrm{s}_0(u-\nu_\ell)\,\mathrm{s}_2(u+\nu_\ell)}\,.\end{aligned} \tag{C6}$$

Here $\mathfrak{K}^\pm_{\alpha,\delta}(u)$ label the diagonal entries (i.e. the eigenvalues) of the diagonal boundary matrices,

$$\begin{aligned}\mathfrak{K}_\alpha^-(u) &= \omega^-\sin(\psi_- + u)\ ,\ \mathfrak{K}_\alpha^+(u) = \omega^+\sin(u + 2\eta + \psi_+)\ ,\\ \mathfrak{K}_\delta^-(u) &= \omega^-\sin(\psi_- - u)\ ,\ \mathfrak{K}_\delta^+(u) = \omega^+\sin(u + 2\eta - \psi_+)\ .\end{aligned} \tag{C7}$$

[6] Note, that this result corresponds to the one obtained by [13]. However, the authors of [13] seem to have made a slight mistake when substituting their formula (57) into (61) to obtain (62), which should correctly read

$$t(u) = +\frac{\sin(2u+4\eta)\sin(u+t^+)}{\sin(2u+2\eta)}\mathcal{A}(u) - \frac{\sin(u+2\eta-t^+)}{\sin(2u+2\eta)}\tilde{\mathcal{D}}(u)\,.$$

Introducing the functions

$$q(u) \equiv \prod_{\ell=1}^{M} \sin(u+2\eta+\nu_\ell)\sin(u-\nu_\ell) \tag{C8}$$

the eigenvalues (C6) can be recast as

$$\begin{aligned} \Lambda(u)q(u) &= \mathfrak{K}^-_\alpha(u)\left(\mathfrak{K}^+_\alpha(u) - \frac{\mathrm{s}_2(0)}{\mathrm{s}_2(2u)}\mathfrak{K}^+_\delta(u)\right)\left(\frac{\mathrm{s}_2^2(u)}{\mathrm{s}_2(u)\,\mathrm{s}_2(-u)}\right)^N q(u-2\eta) \\ &- \mathfrak{K}^+_\delta(u)\left(\mathfrak{K}^-_\delta(u) - \frac{\mathrm{s}_2(0)}{\mathrm{s}_2(2u)}\mathfrak{K}^-_\alpha(u)\right)\left(\frac{\mathrm{s}_0^2(u)}{\mathrm{s}_2(u)\,\mathrm{s}_2(-u)}\right)^N q(u+2\eta)\,. \end{aligned} \tag{C9}$$

Appendix D: Super quantum determinants

Consider a generic $BFFB$ graded R-matrix of the shape

$$R(u) = \begin{pmatrix} a(u+2\eta) & 0 & 0 & 0 \\ 0 & a(u) & a(2\eta) & 0 \\ 0 & a(2\eta) & a(u) & 0 \\ 0 & 0 & 0 & -a(u+2\eta) \end{pmatrix} \tag{D1}$$

where $a(-u) = a(u)$ and $a(0) = 0$. At $u = -2\eta$ such an R-matrix gives rise to a projector P^- onto a one-dimensional subspace

$$P^- = -\frac{1}{2a(2\eta)}R(-2\eta) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/2 & -1/2 & 0 \\ 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{D2}$$

Let $T(u)$ be a representation of the graded YBA

$$R_{12}(u-v)\; T_1(u)\; T_2(v) = T_2(v)\; T_1(u)\; R_{12}(u-v) \tag{D3}$$

with the usual embeddings $T_1(u) \equiv T(u) \otimes_s \mathbb{1}$ and $T_2(v) \equiv \mathbb{1} \otimes_s T(v)$, where

$$T(u) \equiv \begin{pmatrix} T^1{}_1(u) & T^1{}_2(u) \\ T^2{}_1(u) & T^2{}_2(u) \end{pmatrix}_{BF} \equiv \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}_{BF}. \tag{D4}$$

The PBC super quantum determinant (SQD) is defined as

$$\delta\{T(u)\} \equiv \mathrm{str}_{12}\{ P^-_{12}\; T_1(u)\; T_2(u+2\eta)\} \tag{D5}$$

$$\begin{aligned} &= \frac{1}{2}\{C(u)B(u+2\eta) - A(u)D(u+2\eta) \\ &\quad -B(u)C(u+2\eta) - D(u)A(u+2\eta)\}\,. \end{aligned} \tag{D6}$$

At $v = u + 2\eta$ and after dividing by $a(2\eta)$ the graded YBA yields the commutation relations

$$C(u)B(u+2\eta) - A(u)D(u+2\eta) = C(u+2\eta)B(u) - D(u+2\eta)A(u) \tag{D7}$$

$$D(u)A(u+2\eta) + B(u)C(u+2\eta) = D(u+2\eta)A(u) - C(u+2\eta)B(u) \tag{D8}$$

$$B(u)C(u+2\eta) + D(u)A(u+2\eta) = B(u+2\eta)C(u) + A(u+2\eta)D(u) \tag{D9}$$

These relations can be used to simplify the super quantum determinant to

$$\delta\{T(u)\} = -[A(u)D(u+2\eta) - C(u)B(u+2\eta)] \ . \tag{D10}$$

It remains to show that the super quantum determinant is a central element of the graded YBA, i.e. that it supercommutes with all the other elements $A(v)$, $B(v)$, $C(v)$ and $D(v)$ for arbitrary v. Consider the expression

$$R_{12}(u-v)R_{13}(u-w)R_{23}(v-w)T_1(u)T_2(v)T_3(w) \ . \tag{D11}$$

Employing the graded YBE once it is obvious that

$$\begin{aligned} \text{(D11)} \ &= \ R_{23}(v-w)R_{13}(u-w) \ \ [R_{12}(u-v)T_1(u)T_2(v)] \ \ T_3(w) &\text{(D12)}\\ \boxed{v \to u+2\eta} \ &\Rightarrow \ -2a(2\eta)R_{23}(u-w+2\eta)R_{13}(u-w)P^-_{12}T_1(u)T_2(u+2\eta)T_3(w) \ . &\text{(D13)} \end{aligned}$$

On the other hand, by applying the graded YBA relation twice it is found that

$$\begin{aligned} \text{(D11)} \ &= \ T_3(w) \ \ [R_{12}(u-v)T_1(u)T_2(v)] \ \ R_{13}(u-w)R_{23}(v-w) &\text{(D14)}\\ \boxed{v \to u+2\eta} \ &\Rightarrow \ -2a(2\eta)T_3(w)P^-_{12}T_1(u)T_2(u+2\eta)R_{13}(u-w)R_{23}(u-w+2\eta) \ . &\text{(D15)} \end{aligned}$$

Equating (D13) and (D15) and multiplying from both sides with P^-_{12} gives

$$\begin{aligned} &\{P^-_{12}R_{23}(u-w+2\eta)R_{13}(u-w)P^-_{12}\}\{P^-_{12}T_1(u)T_2(u+2\eta)P^-_{12}\}T_3(w) \\ &= T_3(w)\{P^-_{12}T_1(u)T_2(u+2\eta)P^-_{12}\}\{P^-_{12}R_{13}(u-w)R_{23}(u-w+2\eta)P^-_{12}\} \end{aligned} \tag{D16}$$

where additional P^-_{12} projectors have been inserted by virtue of the appropriate triangularity conditions. After a change of basis to the P^-_{12} eigenbasis via A_{12} as defined in (3.14), it is easy to check that application of the supertrace $\mathrm{str}_{12}\{\,.\,\}$ yields

$$\begin{aligned} &\sigma^z_3 \ \delta\{T(u)\} \ \ T_3(w) &= T_3(w) \ \sigma^z_3 \ \delta\{T(u)\} \qquad &\text{(D17)}\\ \Leftrightarrow \quad &[\sigma^z_3 \ \delta\{T(u)\}, T_3(w)] &= 0 \qquad &\text{(D18)}\\ \Leftrightarrow \quad &\Big[\delta\{T(u)\}, T^i{}_j(w)\Big]_\pm &= 0 \ . \qquad &\text{(D19)} \end{aligned}$$

Similarly one may introduce the object

$$\delta\left\{\hat{T}(u)\right\} \equiv \mathrm{str}_{12}\left\{ P_{12}^{-}\ \hat{T}_2(u)\ \hat{T}_1(u+2\eta) \right\} \tag{D20}$$

which obeys the exact same super commutation relations and by (6.9) turns out to be proportional to the inverse of the above SQD. In particular for the considered N-site small polaron model it is found that

$$\begin{aligned} \delta(u) \ &\equiv\ \delta\left\{T(u)\right\} = -\zeta^N(u+2\eta)\ \prod_{i=1}^{N}(-\sigma_{q_i}^z) && \text{(D21a)} \\ \hat{\delta}(u) \ &\equiv\ \delta\{\hat{T}(u)\} = -\frac{1}{\zeta^N(u)}\ \prod_{i=1}^{N}(-\sigma_{q_i}^z) && \text{(D21b)} \end{aligned}$$

where q_i labels the i-th quantum subspace (cf. section V B). Moreover, the commutation relation (D18) extends to the fused quantities according to

$$[\sigma^z_{\ll n \gg}\ \delta\left\{T(u)\right\}, T_{\ll 1\ldots n\gg}(w)] = 0 \tag{D22}$$

with $\sigma^z_{\ll n\gg}$ being defined in equation (3.16).

In the open boundary case, the place of $\delta\left\{T(u)\right\}$ is taken by another object $\Delta(u)$ which will most appropriately be called the OBC super quantum determinant. Generally, the SQD is what you get when you alter the first fusion step such that, instead of creating a *higher dimensional* transfer matrix by projection on a three dimensional auxiliary space, you now create a *lower* dimensional object by projecting onto the complementary one dimensional space. In a sense, loosely speaking, you do a reduction instead of a fusion and find that the open boundary SQD factors as follows,

$$\begin{aligned} \Delta(u) &\equiv \mathrm{str}_{12}\left\{ P_{12} K_2^{+}(u+2\eta)\bar{R}_{12}(-2u-6\eta)K_1^{+}(u)\mathcal{T}_1^{-}(u)R_{12}(2u+2\eta)\mathcal{T}_2^{-}(u+2\eta) \right\} \\ &= \delta\left\{K^{+}(u)\right\}\cdot\delta\left\{T(u)\right\}\cdot\delta\left\{K^{-}(u)\right\}\cdot\delta\{\hat{T}(u)\} \\ &= \left(\frac{\zeta(u+2\eta)}{\zeta(u)}\right)^N \delta\left\{K^{+}(u)\right\}\cdot\delta\left\{K^{-}(u)\right\} \end{aligned} \tag{D23}$$

where $\mathcal{T}^{-}(u)$ was defined in (6.8) and

$$\begin{aligned} \delta\left\{K^{+}(u)\right\} &\equiv \mathrm{str}_{12}\left\{ P_{12}^{-}\ K_2^{+}(u+2\eta)\ \bar{R}_{12}(-2u-3\cdot 2\eta)\ K_1^{+}(u) \right\} \\ &= g(-2u-6\eta)\cdot\det\left\{K^{+}(u)\right\} \end{aligned} \tag{D24a}$$

$$\begin{aligned} \delta\left\{K^{-}(u)\right\} &\equiv \mathrm{str}_{12}\left\{ P_{12}^{-}\ K_1^{-}(u)\ R_{21}(2u+2\eta)\ K_2^{-}(u+2\eta) \right\} \\ &= g(2u+2\eta)\cdot\det\left\{K^{-}(u+2\eta)\right\} \end{aligned} \tag{D24b}$$

with the function $g(u)$ being introduced in the context of (2.5d). Since $\alpha_\pm\cdot\beta_\pm = 0$, as mentioned in section VI A, the expressions $\det\left\{K^{\pm}(u)\right\}$ are well defined.

Appendix E: Transformation matrices

This appendix presents a collection of matrix representations of the various similarity transformations employed in this paper. It is convenient to define the coefficients

$$a_n \equiv \sqrt{\frac{2n}{n+1}}\ \left([n]_q\big|_{\eta=\eta_n}\right)^{-1/2} \quad \text{and} \quad b = \left(\frac{[2]_q\big|_{\eta=\eta_2}}{[3]_q\big|_{\eta=\eta_3}}\right)^{-1/2} \tag{E1a}$$

where $[n]_q$ denotes the usual q-deformation of an integer $n \in \mathbb{N}$ defined by

$$[n]_q \equiv \frac{q^n - q^{-n}}{q - q^{-1}} \quad \text{with} \quad q \equiv \mathrm{e}^{2\mathrm{i}\eta} \tag{E1b}$$

and to set

$$A_{(1)} \equiv B_{\ll 1 \gg} \equiv C_{\ll 1 \gg} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} . \tag{E1c}$$

$$A_{(12)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \tag{E2a}$$

$$B_{\ll 12 \gg} = \mathrm{diag}(a_2, 1, a_2) \tag{E2b}$$

$$C_{\ll 12 \gg} = \mathrm{diag}(a_2, 1, 1) \tag{E2c}$$

$$A_{(123)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{2\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} & 0 & -\frac{\sqrt{2}}{3} & 0 & 0 & 0 \\ 0 & -\frac{\sqrt{2}}{3} & \frac{2\sqrt{2}}{3} & 0 & -\frac{\sqrt{2}}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2\sqrt{2}}{3} & 0 & -\frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} & 0 \\ 0 & 0 & 0 & -\frac{\sqrt{2}}{3} & 0 & \frac{2\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} & 0 \end{pmatrix} \tag{E3a}$$

$$B_{\ll 12 \gg} = \mathrm{diag}(a_3, 1, 1, a_3) \tag{E3b}$$

$$C_{\ll 123 \gg} = \mathrm{diag}(a_3, 1, 1, 1) \tag{E3c}$$

$$A_{(1234)} = \frac{1}{\sqrt{2}} \begin{pmatrix}
\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} \\
0 & \frac{3}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{2} & \frac{3}{2} & 0 & -\frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{5}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & 0 & 0 & 0 \\
0 & -\frac{1}{2} & -\frac{1}{2} & 0 & \frac{3}{2} & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{3} & 0 & \frac{5}{3} & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{3} & 0 & -\frac{1}{3} & \frac{5}{3} & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{2} & 0 & 0 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 \\
0 & 0 & 0 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & \frac{5}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & \frac{5}{3} & 0 & -\frac{1}{3} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & \frac{3}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} & 0 & \frac{3}{2} & -\frac{1}{2} & 0
\end{pmatrix} \tag{E4a}$$

$$B_{\ll 1234 \gg} = \mathrm{diag}(a_4, 1, b, 1, a_4) \tag{E4b}$$

$$C_{\ll 1234 \gg} = \mathrm{diag}(a_4, 1, b, 1, 1) \tag{E4c}$$

[1] V. K. Fedyanin and L. V. Yakushevich, Theor. Math. Phys. **37**, 1081 (1978), translated from Teoreticheskaya i Matematicheskaya Fizika, Vol. 37 (1978) 371-381.

[2] V. G. Makhankov and V. K. Fedyanin, Phys. Rep. **104**, 1 (1984).

[3] V. Korepin, N. Bolgoliubov, and A.G.Izergin, *Quantum Inverse Scattering Method and Correlation Functions* (Cambridge University Press, 1993).

[4] F.-C. Pu and B.-H. Zhao, Phys. Lett. A **118**, 77 (1986).

[5] H.-Q. Zhou, L.-J. Jiang, and P.-F. Wu, Phys. Lett. A **137**, 244 (1989).

[6] L. Mezincescu and R. I. Nepomechie, J. Phys. A **24**, L17 (1991).

[7] A. González-Ruiz, Nucl. Phys. B **424**, 468 (1994).

[8] A. J. Bracken, X.-Y. Ge, Y.-Z. Zhang, and H.-Q. Zhou, Nucl. Phys. B **516**, 588 (1998), cond-mat/9710141.

[9] E. K. Sklyanin, J. Phys. A: Math. Gen. **21**, 2375 (1988).

[10] H.-Q. Zhou, J. Phys. A **29**, L607 (1996).

[11] H.-Q. Zhou, J. Phys. A **30**, 711 (1997).

[12] X.-W. Guan, U. Grimm, and R. A. Roemer, Ann. Phys. (Leipzig) **7**, 518 (1998), cond-mat/9811089.

[13] Y. Umeno, H. Fan, and M. Wadati, J. Phys. Soc. Japan **68**, 3826 (1999), cond-mat/9907474.

[14] X.-W. Guan, H. Fan, and S.-D. Yang, Phys. Lett. A **251**, 79 (1999).

[15] X.-M. Wang, H. Fan, and X.-W. Guan, J. Phys. Soc. Japan **69**, 251 (2000), see also H. Fan and X.-W. Guan, cond-mat/9711150.

[16] J. F. Cornwell, *Group Theory in Physics*, vol. III - Supersymmetries and Infinite-Dimensional Algebras (Academic Press, 1989).

[17] R. I. Nepomechie, Nucl. Phys. B **622**, 615 (2002), hep-th/0110116.

[18] J. Cao, H.-Q. Lin, K.-J. Shi, and Y. Wang, Nucl. Phys. B **663**, 487 (2003), cond-mat/0212163.

[19] R. I. Nepomechie, J. Phys. A **37**, 433 (2004), hep-th/0304092.

[20] W.-L. Yang, R. I. Nepomechie, and Y.-Z. Zhang, Phys. Lett. B **633**, 664 (2006), hep-th/0511134.

[21] P. Baseilhac and K. Koizumi, J. Stat. Mech. p. P09006 (2007), hep-th/0703106.

[22] W. Galleas, Nucl. Phys. B **790**, 524 (2008), 0708.0009.

[23] H. Frahm, A. Seel, and T. Wirth, Nucl. Phys. B **802**, 351 (2008), 0803.1776.

[24] H. Frahm, J. R. Grelik, A. Seel, and T. Wirth, J. Phys. A **44**, 015001 (2011), 1009.1081.

[25] G. Niccoli, preprint (2012), 1206.0646.

[26] A. M. Grabinski and H. Frahm, J. Phys. A: Math. Theor. **43**, 045207 (2010), 0910.4029.

[27] R. Murgan and R. I. Nepomechie, J. Stat. Mech. p. P05007 (2005), hep-th/0504124.

[28] R. Murgan, R. I. Nepomechie, and C. Shi, J. Stat. Mech. p. P08006 (2006), hep-th/0605223.

[29] R. Murgan, R. I. Nepomechie, and C. Shi, JHEP **0701**, 038 (2007), hep-th/0611078.

[30] F. Göhmann and S. Murakami, J. Phys. A **31**, 7729 (1998), cond-mat/9805129.